Nonlinear Dynamic Factor Models

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Motivation

Linear factor models intensively used in macroeconomics do not capture nonlinearities arising during deep recessions – financial crisis 2008/2009 – or binding constraints – ZLB.

Dynamic Factor Model

We propose a nonlinear dynamic factor model featuring a state equation pruned to the second order based on Andreasen et al. (2013) and a general nonlinear measurement equation.

Implementation and Applications

We use Uscented Kalman Filter and Particle Filter for Maximum Likelihood estimation of the model on US key macroeconomic indicators and cross-country panel of CDS spreads data.

Model

General model:

$$\text{Measurement } y_t = \mathcal{G}(f_t) + \eta\epsilon_t$$
$$\text{Factor motion } f_t = \mathcal{H}(f_{t-1}) + \sigma\nu_t.$$

Here, ϵ_t and ν_t are $iid\ N(0, I)$. Using a generic nonlinear factor motion may lead to

1. Explosive dynamics
2. Divergence of filter

To fix this, we use pruning (Andreasen et al., 2013) – it allows to limit higher order effects. The model is – up to second-order effects:

$$\begin{cases} f_t = f_t^1 + f_t^2 \text{ – 1st and 2nd order factors} \\ f_t^1 = \mathcal{H}_x f_{t-1}^1 + \sigma\nu_t \\ f_t^2 = \mathcal{H}_x f_{t-1}^2 + 0.5\mathcal{H}_{xx}\left(f_{t-1}^1 \times f_{t-1}^1\right). \end{cases} \quad (1)$$

At this point, we don't impose any restrictions on $\mathcal{G}(\cdot)$. Furthermore, our approach can be easily extended to more factors, alternative distributional forms – heteroskedasticity, kurtosis, skewness.

Unscented Kalman Filter – UKF

Approach: Use unscented transform for approximating filtering distributions. UKF forms a Gaussian approximation to the filtering distribution:

$$p(f_t|Y_{1:t}) \approx N(f_t|m_t, P_t).$$

Here, m_t and P_t are mean and covariance. UKF captures first and second moments of the resulting random variables using sigma points.
Algorithm from Särkkä (2013):

- **Prediction**
  1. Given m_{t-1} and P_{t-1}, form sigma points:
  $$\chi_{t-1}^{(0)} = \mathbf{m}_{t-1}, \chi_{t-1}^{(i)} = \mathbf{m}_{t-1} + \sqrt{n+\lambda}\left[\sqrt{\mathbf{P}_{t-1}}\right]_i, \chi_{t-1}^{(i+n)} = \mathbf{m}_{t-1} - \sqrt{n+\lambda}\left[\sqrt{\mathbf{P}_{t-1}}\right]_i,$$
  for $i = 1, \dots, n$,. Here $\lambda = \alpha^2(n+t) - n$ (n – dimensionality of the state), $\mathbf{m}$ is filtered state's (factor's) mean, $\mathbf{P}$ – filtered state's covariance matrix, t denotes period.
  2. Propagate sigma points through the dynamic model:
  $$\hat{\chi}_t^{(i)} = \mathcal{H}(\chi_{t-1}^{(i)}), i = 0, \dots, 2n$$
  3. Compute predicted mean $\mathbf{m}_t^-$ and predicted covariance $\mathbf{P}_t^-$
  $$\mathbf{m}_t^- = \sum_{i=0}^{2n} W_i^{(m)}\hat{\chi}_t^{(i)}, \quad \mathbf{P}_t^- = \sum_{i=0}^{2n} W_i^{(c)}\left(\hat{\chi}_t^{(i)} - \mathbf{m}_t^-\right)\left(\hat{\chi}_t^{(i)} - \mathbf{m}_t^-\right)^T + \mathbf{Q}_{t-1},$$
- **Update step**
  1. Form the sigma points:
  $$\chi_{t-1}^{-(0)} = \mathbf{m}_{t-1}^-, \chi_{t-1}^{-(i)} = \mathbf{m}_{t-1}^- + \sqrt{n+\lambda}\left[\sqrt{\mathbf{P}_{t-1}^-}\right]_i, \chi_{t-1}^{-(i+n)} = \mathbf{m}_{t-1}^- - \sqrt{n+\lambda}\left[\sqrt{\mathbf{P}_{t-1}^-}\right]_i$$
  2. Propagate sigma points through the measurement model:
  $$\hat{y}_t^{(i)} = \mathcal{G}\left(\chi_t^{-(i)}\right), i = 0, \dots, 2n$$
  3. Compute predicted mean μ_t, predicted covariance of the measurement $\mathbf{S}_t$:
  $$\mu_t = \sum_{i=0}^{2n} W_i^{(m)}\hat{\mathcal{Y}}_t^{(i)}, \quad \mathbf{S}_t = \sum_{i=0}^{2n} W_i^{(c)}\left(\hat{\mathcal{Y}}_t^{(i)} - \mu_t\right)\left(\hat{\mathcal{Y}}_t^{(i)} - \mu_t\right)^T + \mathbf{R}_t$$

Advantage: Computationally fast
Disadvantage: Captures only 1st two moments of the distribution
Implementation: Maximum likelihood estimation based on Särkkä (2013) Matlab package

Particle Filter – PF

Approach: generates distribution of particles, weights each particle according to its likelihood, resamples to avoid degeneration and propagates through nonlinear system.
Bootstrap filter version from Särkkä (2013):

- **Prediction**
  Given particles and weights at $t-1$: $\{x_{t-1}^i, W_{t-1}^i\}$
  1. Draw a new particle $x_t^{(i)}$ for each point in the sample set $\{x_{t-1}^{(i)} : i = 1, \dots, N\}$ from
  $$x_t^{(i)} \sim p(x_t|x_{t-1}^{(i)}), \quad i = 1, \dots, N$$
  2. Calculate weights:
  $$\omega_t^{(i)} = p(y_t|x_t^{(i)}), \quad i = 1, \dots, N$$
- **Update**
  1. Define normalized weights: $\widetilde{W}_t^{(i)} = \frac{\omega_t^{(i)}W_{t-1}^{(i)}}{\frac{1}{N}\sum\omega_t^{(i)}W_{t-1}^{(i)}}$.
  2. Resample from multinomial distribution $\left\{\omega_t^{(i)}, \widetilde{W}_t^{(i)}\right\}$ and set $W_t^{(i)} = 1$.

Approximate state distribution and likelihood are:

$$p(x_t|Y_{1:t}) \approx \sum_{i=1}^{N}\omega_t^{(i)}\delta(x_t - x_t^{(i)}), \quad p(y_t|Y_{1:t-1}) \approx \frac{1}{N}\sum_{i=1}^{N}\omega_t^i W_{t-1}^i. \quad (2)$$

Advantage: Tracks the whole distribution
Disadvantage: Computationally and coding-wise heavy
Implementation: Maximum likelihood estimation using CUDA/C++

US Macroeconomic indicators with UKF

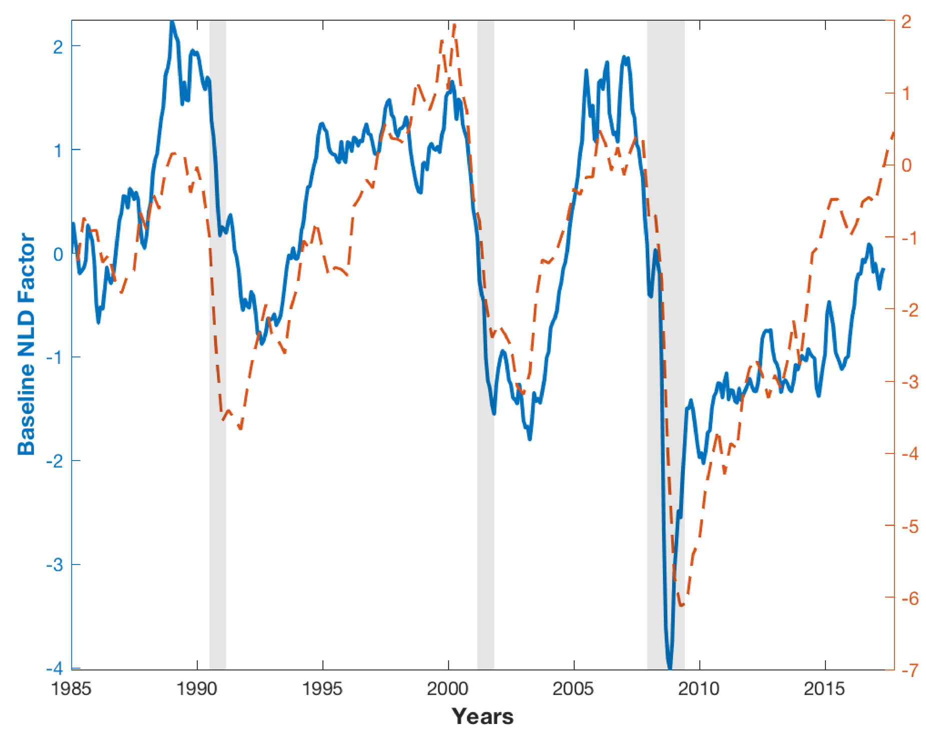
Measurement equation:

$$\begin{bmatrix} R_t \\ y_t \end{bmatrix} = \begin{bmatrix} \max(g_r f_t, -\frac{\mu_r}{\sigma_r}) \\ G \times f_t \end{bmatrix} + \eta\epsilon_t$$

where $\epsilon_t \sim iidN(0, \mathbf{I})$, f follows (1) and $\nu_t \sim iidN(0, 1)$, R is Fed funds rate, y_t includes hourly earnings* , spread between Baa corporate bond yield and 10-year Treasury, CPI inflation*, industrial production index*, spread between 10-Year Treasury Constant Maturity and 2-Year Treasury Constant Maturity, and weekly hours worked, all the series obtained from FRED.

Monthly data covering 1985:1 - 2017:6. Series marked with * were used in log differences $x_t = \ln(\tilde{x}_t) - \ln(\tilde{x}_{t-1})$, where $\tilde{x}$ are original non-stationary series, and all series were standardized: $y_t = \frac{x_t - \bar{x}}{\sigma(x)}$, where x_t are original or log-differentiated series, $\bar{x}$ denotes average and $\sigma(x)$ – standard error. UKF parameters: $n = 2$ (5 sigma points), $\alpha = 1,\ \beta = 0,$ and $\kappa = 1$.

Figure 1. Filtered factor in the estimated model and Output Gap. Data source: FRED, CBO, NBER



Results: The resulting nonlinear economic activity index tracks closely the CBO's output gap.

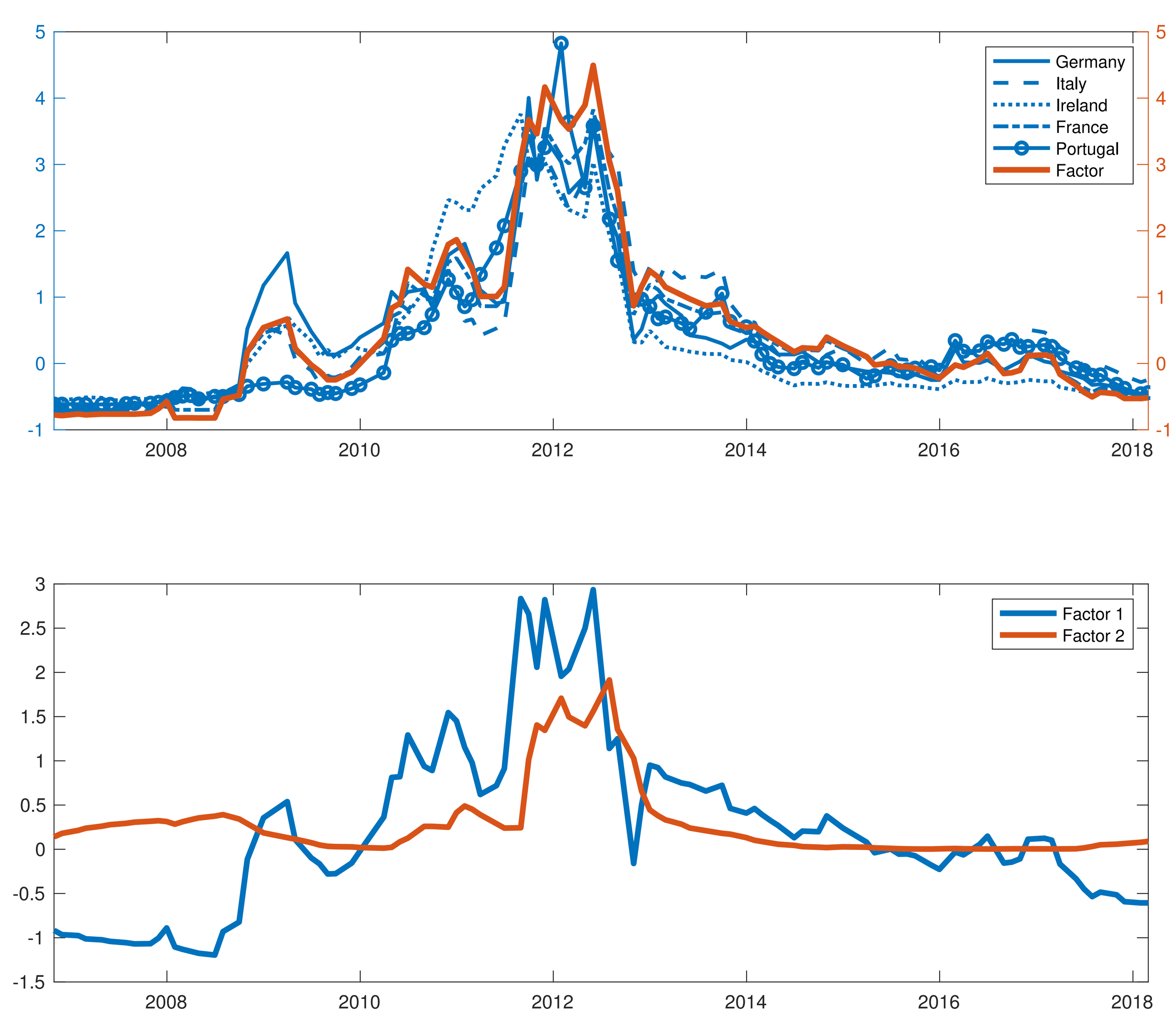
European CDS spreads with PF

Measurement equation:

$$y_t = G \times f_t + \eta\epsilon_t,$$

where y_t are CDS spreads, G is a **5x1** vector, $\epsilon_t \sim iidN(0, \mathbf{I})$ and factor evolves following (1) . We use 5-year USD CDS end of the month spreads obtained from Thomson Reuters Datastream for Germany, Italy, Ireland, France, and Portugal. Sample covers: 2006:10 - 2018:2. All the series were standardized: $y_t = \frac{x_t - \bar{x}}{\sigma(x)}$, where x_t are original series, $\bar{x}$ denotes average and $\sigma(x)$ – standard error. We use $10,000$ particles.

Figure 2. Factor and its decomposition in standardized EU CDS spread series. Data source: Thomson Reuters Datastream



Results: We uncover a common factor that reflects common default risk within the sample of European economies. "Second-order" component of this factor demonstrates significant fluctuations in the period of the sovereign debt crisis, just moderately varying in the rest of the available time interval.

Bibliography

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