EUROPEAN CENTRAL BANK

EUROSYSTEM

Working Paper Series

Elena Carletti, Agnese Leonello, Robert Marquez

Loan guarantees, bank underwriting policies and financial fragility

No 2782 / February 2023

Abstract

Loan guarantees represent a form of government intervention to support bank lending. However, their use raises concerns as to their effect on bank risk-taking incentives. In a model of financial fragility that incorporates bank capital and a bank incentive problem, we show that loan guarantees reduce depositor runs and improve bank underwriting standards, except for the most poorly capitalized banks. We highlight a novel feedback effect between banks' underwriting choices and depositors' run decisions, and show that the effect of loan guarantees on banks' incentives is different from that of other types of guarantees, such as deposit insurance.

Keywords: panic runs, fundamental runs, bank monitoring, charter value

JEL classifications: G21, G28

Non-technical summary

The Covid-19 pandemic erupted in early 2020 as an unexpected and exogenous shock leading to a sudden and deep liquidity crisis for non-financial corporations. To minimize disruptions to the real economy, major forms of public interventions across countries were implemented. Among those, public guarantee schemes (PGSs), which aimed at sustaining bank lending to firms by providing a guarantee on bank loans, played a major role.

Notwithstanding their possible effectiveness as a stimulative tool, the use of loan guarantees raises several important questions in terms of their implications for banks' underwriting processes and thus, ultimately, for financial stability. The analysis of the incentive effects of loan guarantees and their implications for financial fragility is precisely the focus of this paper.

We tackle these issues by developing a theoretical framework where both banks' risk choices on the asset side and financial fragility are derived endogenously and, thus, a feedback effects between bank lending decisions and investors' behavior is present. As is well known, bank choices on the asset side also crucially depend on their capital structures. In particular, the degree of bank capitalization as well as the possibility for investors to withdraw their funds has been found to affect bank lending decisions (see, e.g., the evidence in Iyer and Puri, 2012; Iyer, Puri and Ryan, 2016; Martin, Puri and Ufier, 2018; Artavanis, Paravisini, Robles-Garcia, Seru and Tsoutsoura, 2019; or Carletti, De Marco, Ioannidou and Sette, 2020). This implies that the quality of a bank's assets, the threat of runs, and its capital structure are closely intertwined.

Our model of financial fragility is in the spirit of Goldstein and Pauzner (2005), which we enrich in two important dimensions. First, we assume that banks maximize profits, and fund themselves with equity in addition to demandable deposits. Second, we introduce a risk choice for banks by assuming that they can affect the success probability of loans when choosing their underwriting effort. These two aspects allow us to analyze the interaction between the asset and liability side of banks' balance sheet and to stress the importance of bank capital structure for the overall effects of the guarantees in terms of banks' underwriting and financial stability.

We first show that banks are subject to runs, whose probability decreases with the level of bank capitalization. In addition, banks with high levels of capital are subject to runs only when macroeconomic fundamentals are sufficiently poor (fundamental-driven runs), while banks with low capital are also prone to panic runs, meaning that their depositors may decide to run for reasons linked to strategic complementarity problems that arise when they anticipate other depositors may run. It follows that, for any level of capital, banks can only fail in the final period when their underwriting effort turns out to be unsuccessful.

As with any form of insurance, the introduction of a loan guarantee reduces depositors' run probability. The reason is that the guarantee increases the range in which the bank is able to make the promised repayment to depositors in the final date and, if the government transfers are bankruptcy-protected, it also increases depositors' expected payoffs at the final date. Both effects reduce depositors' incentives to withdraw prematurely, thus reducing financial fragility.

In our framework, contrary to perceived wisdom, introducing loan guarantees improves banks' underwriting standards in many instances. This finding arises both from a direct (positive) effect of loan guarantees and an indirect effect from the reduction in run probability. The result may differ only when the loan guarantee is shielded from bankruptcy costs. In this case, the presence of a loan guarantee reduces the sensitivity of the run threshold to changes in the underwriting effort. This last

effect is negative as it reduces the benefit for the bank from increasing its effort, and it may dominate when banks are insufficiently capitalized.

One crucial element of the analysis is whether the guarantee accrues to the bank conditional on its ability to control risk. In the model, the guarantee is disbursed whenever the firm is unable to repay the bank. However, whether the bank or its creditors benefit from the guarantee when the bank's underwriting effort is successful (i.e., when positive project returns are realized) depends on the treatment of the guarantee in bankruptcy. In the case of full bankruptcy costs, both the bank and depositors only stand to receive anything if there is no run and the bank remains solvent. By contrast, in the case of bankruptcy-protected guarantees, depositors also receive some payment in the final period when the bank's underwriting effort is unsuccessful and the bank defaults. This reduces the sensitivity of depositors' incentives to run to the bank's underwriting standards, thus indirectly benefiting the bank and reducing its incentives. Therefore, the treatment of the guarantee in bankruptcy becomes de facto equivalent to a conditionality assumption.

We also use our framework to study the effect of PGSs on banks' evergreening incentives (i.e., continue projects that would be efficient to liquidate). To do so, we extend the analysis to include banks' project continuation decisions by allowing them to liquidate projects at the interim date. In the absence of runs, all banks would engage in evergreening and, in line with the empirical evidence (see e.g., Blattner, Farinha and Rebelo, 2021; and Schivardi, Sette and Tabellini, 2021), the more so the lower is their level of capitalization. Once runs are taken into account, loans can be liquidated early either because of runs or directly by the bank. For banks with low capital, depositors exert a strong disciplinary force and projects get liquidated early because of depositor runs. By contrast, when banks have high capital, depositors are more passive and early liquidation occurs primarily as a result of banks' decisions. In this context, the introduction of loan guarantees leads to more evergreening since depositors' incentives to run decrease, while banks' incentives to continue inefficient projects increase, in particular for worse-capitalized banks.

1 Introduction

Periods of crisis, when economic fundamentals are poor, are catalysts for government intervention. Often these periods are coupled with credit market freezes, with banks sitting on capital rather than lending it out, possibly further worsening fundamentals to the extent that viable firms get denied credit. A case in point is the Covid-19 pandemic, which erupted in early 2020 as an unexpected shock leading to a sudden and deep liquidity crisis for non-financial corporates and triggering massive interventions by public authorities (e.g., Eichenbaum, Rebelo, and Trabandt, 2020; Ding, et al., 2020; Li, Strahan, and Zhang, 2020).

Despite differences across countries, one major form of intervention consisted of public guarantee schemes (PGSs) on loans aimed at supporting the flow of credit to the economy following the decline in economic fundamentals. As described in more detail in Section 2 below, one important element in common in these schemes was their use as stimulative tools through the offer of credit protection against the default of the borrower. While sharing the objectives of such programs, their widespread use has also raised important questions among economists concerning their impact on lending standards and continuation decisions. While PGSs are typically administered by a public authority, the final lending decisions (i.e., selection and monitoring of the recipient) remain in fact with the financial intermediary. It follows that, as with any form of insurance, the introduction of PGSs may generate moral hazard by encouraging riskier lending at the margin through banks' reduced incentives to select and monitor borrowers properly (e.g., Kelly, Lustig, and Van Nieuwerburgh, 2016; Gropp, Gruendl and Guettler, 2014).[1] Similarly, the reliance on public support programs may induce banks, and in particular those with little capital, to engage in "evergreening," thus keeping nonviable firms alive (see, e.g., Acharya et al., 2020a; Acharya et al., 2020b; Laeven, Schepens, and Schnabel, 2020, Dursun-de Neef and Schandlbauer, 2021).

In this paper, we analyze the effect of loan guarantees on banks' underwriting incentives in a framework where the asset and liability sides of banks' balance sheets endogenously interact and jointly determine banks' incentives. Unlike other guarantee schemes, such as deposit insurance, loan guarantees accrue to banks and help them remain solvent, thus having the potential to directly influence bank behavior. In addition, they can also benefit depositors when losses would put the bank at risk of defaulting on its liabilities, and therefore they have implications for depositors' withdrawal behavior. Evidence supports the incentive view of demandable debt (e.g., Iyer and Puri,

[1] This is a commonly held view concerning the impact of deposit insurance, for instance, and is often cited as a rationale for prudential policies (e.g., capital requirements) to control excessive risk-taking.

2012; Iyer, Puri, and Ryan, 2016; Martin, Puri, and Ufier, 2018; Artavanis et al., 2019; and Carletti et al., 2020): investors react to signals on banks' fundamentals when deciding whether to withdraw their funds and, anticipating this, banks take investors' reactions into account when making their lending decisions. It follows that the impact of loan guarantees for lending is best analyzed in a framework that incorporates the feedback between bank lending decisions and depositor withdrawal decisions.

To this end, we present a model of financial fragility in the spirit of Goldstein and Pauzner (2005), which we enrich along two important dimensions. First, we introduce an endogenous effort problem so that, through their underwriting decisions, banks can influence the success probability of the loans they extend. Second, we assume that banks maximize profits and fund themselves with equity in addition to deposits.[2]

The model has two periods. In the first period, banks with some equity capital raise additional funds in the form of (demandable) debt and grant long-term loans to finance firms' projects. These projects yield a return in the final period that depends on both bank effort and economic fundamentals. Depositors may leave their funds in the bank until projects mature or they may withdraw in the first period, thus precipitating a run. As is common in global-game models of bank runs, depositors base their withdrawal decisions on a signal they receive in an intermediate period, which provides them with information on the fundamentals and which allows them to draw inferences on other depositors' behavior. If the bank is unable to meet its obligations at the final date, the bank's default leads to costly bankruptcy.

We first show that banks are subject to runs with a probability that decreases with the amount of capital they have. Highly capitalized banks are subject to runs only when macroeconomic fundamentals are sufficiently poor, while banks with less capital are also prone to panic runs, meaning that their depositors may run because of coordination failures among them. This role for bank capital in determining banks' exposure to depositor panics is reminiscent of Diamond and Rajan (2000), who argue that capital reduces the cost of excessive runs. Anticipating depositors' withdrawal decisions, banks set the long term payoff on the deposit contract as well as their underwriting standards.

We then analyze the introduction of loan guarantees. We focus on a scheme in which the government is in a first-loss position so that, whenever a borrower fails to repay the promised

[2] Thus, our framework features both an endogenous run probability and an endogenous bank effort choice, extending the analysis in Calomiris and Kahn (1991) and Diamond and Rajan (2000 and 2001), where the threat of a run plays a disciplinary role for bank incentives together with own capital.

amount, the government makes a transfer to the bank to cover the loss, up to some limit. Key for the analysis is the treatment of such transfer in case of bank default. We consider two cases: either the transfer is subject to the same bankruptcy costs as any other bank asset or it is protected from such costs and is not subject to dissipation. The two cases reflect different views on the nature of bankruptcy costs. The former case, which we refer to as “full bankruptcy costs," reflects a situation where bankruptcy losses originate primarily from inefficiencies in bankruptcy procedures due to hold-up problems among creditors or inefficient judicial systems. The latter, which we denote as "bankruptcy-protected," captures instead a setting where bankruptcy losses stem primarily from the illiquidity of bank assets, such as loans, and hence do not apply to more liquid assets such as government transfers.

Loan guarantees allow banks to obtain higher profits when they are solvent and also to repay deposits when losses get sufficiently large. As a result, the presence of a loan guarantee always leads to a reduction in depositors' incentives to run. Combined with banks' increased profits, these two effects together contribute to increasing banks' charter values, thus increasing their incentives to avoid default. This leads to improved underwriting standards when the guarantees are subject to dissipation in the event of bankruptcy. The mechanism is reminiscent of that in Cordella, Dell'Ariccia, and Marquez (2018), where greater deposit guarantees may sometimes lead to better monitoring, and distinct from papers such as Marcus (1984) or Keeley (1990), where changes in charter value are driven by the degree of banking competition.

By contrast, when guarantees are bankruptcy-protected, the introduction of loan guarantees may worsen bank monitoring incentives. Since depositors obtain the guaranteed transfers also when the bank's monitoring effort is unsuccessful, the presence of loan guarantees makes depositors' withdrawal incentives less sensitive to changes in bank underwriting standards. This effect reduces the benefit for banks to exert effort and is dominant for the most poorly capitalized banks.

The impact of loan guarantees on bank underwriting incentives and run probabilities remain qualitatively the same when we consider another guarantee type denoted as a "loss sharing" scheme, in which there is sharing of losses between the government and the bank.[3] However, the two schemes differ in terms of costs and effectiveness for bank incentives. In particular, for a given run probability, the first-loss guarantee provides greater incentives to the bank but at a higher cost.

In a further step, we analyze another form of bank risk-taking in the form of evergreening

[3] The two schemes we consider mirror the structures of the guarantees used in practice in addressing the need for sustaining lending in the aftermath of the Covid-19 pandemic (see, for example, European Commission, 2020).

incentives (i.e., incentives to continue projects that ought to be liquidated), which has also been at the center of the policy discussion. We show that the introduction of loan guarantees leads to more evergreening due to depositors' reduced incentives to run, in particular for worse-capitalized banks. However, the improved underwriting resulting from the loan guarantee partly attenuates this negative effect and on net leads to an increase in total output.

We extend the analysis in two directions. First, we endogenize the deposit rate in the presence of loan guarantees and show that all results remain unaltered. This situation reflects the presence of long standing guarantees such as those used in the US to sustain small businesses. Second, we contrast the results from loan guarantees to those that obtain from deposit insurance. We show that introducing loan guarantees in a context where deposits are insured does not affect the qualitative results concerning bank underwriting standards, although deposit insurance by itself always leads to less bank effort.

2 Public guarantee schemes (PGSs)

Guarantees are relatively common in practice both in private and public forms. For example, Beyhaghi (2021) shows that over one-third of corporate loans issued by US banks are guaranteed by separate legal entities, mostly in the form of personal or corporate guarantors. Similarly, Ahnert and Kuncl (2021) report that 62% of outstanding residential mortgages were insured by the US government through the Government Sponsored Enterprises in 2018.

The goal of PGSs is to improve access to credit for firms, thus also supporting employment. From a policy perspective, loan guarantees can be used as stimulative tools in normal times for businesses that may have difficulty in accessing credit, or they can be a response to sudden shocks weakening economic fundamentals. An example of the former can be found in Small Business Administration (SBA) loans, which provide partial guarantees to private lenders, extending loans to younger firms and supporting employment and credit supply to these firms (e.g., Brown and Earle, 2017; and Bachas, Kim and Yannelis, 2020). An example of the latter can be seen in the response to the outbreak of the Covid-19 pandemic in many economies. To give an order of magnitude, in Europe more than 320 billion euros of new loans were provided under these crises response schemes in just four countries – France, Germany, Italy and Spain – as of September 2020 (ECB, 2020; Falagiarda et al., 2020). Similarly, in the US 5.16 million borrowers had access to guaranteed loans through the $669 billion Paycheck Protection Program (PPP) as of November

2020 (Balyuk, Prabhala, and Puri, 2020; Baudino, 2020; Chodorov-Reich et al., 2021).[4]

Public guarantees are paid when the borrower defaults, thus protecting the lender from credit losses, and can be provided for loans intermediated by different types of lenders. For example, in the US the PPP program was applicable to loans provided by both banks and non-banks such as FinTech lenders, with the two types of lenders extending around 80% and 20%, respectively, of the total guaranteed loans under the program (Howell et al., 2020; Erel and Liebersohn, 2022), in line with the fast-growing importance of FinTech in extending credit to US small businesses (Gopal and Schnabl, 2022). By contrast, in Europe, given the greater reliance on bank lending (e.g., ECB, 2022), the set of lenders eligible for PGSs extended during the pandemic was limited to banks and regulated financial intermediaries (Core and De Marco, 2022).

A growing literature analyzes the role of banks and other lenders as conduits of public liquidity through government guaranteed loans to SMEs in Covid times, both in Europe (e.g., Core and De Marco, 2020; Gonzalez-Uribe and Wang, 2020; Jimenez et al., 2022) and the US (e.g., Balyuk et al., 2020; Bartik et al., 2020; Cole, 2020; Duchin, Martin, and Michaely, 2020; Hubbard and Strain, 2020). The focus in these studies ranges from highlighting the importance of supply heterogeneity in the allocation of guaranteed loans to their implications for firm employment.

While PGSs appear to have been generally successful at maintaining a stable flow of credit, their use as a response to the pandemic has also been viewed as being rather expensive in some circumstances, or of having entailed some fraud or undesired consequences. For example, focusing on the role of banks in the PPP program, Granja et al. (2022) find evidence that the program had little effect on employment in the months following its initial rollout, while Griffin, Kruger, and Mahajan (2022) find that, in the same context, both misreporting and suspicious lending by FinTech companies has increased due to the lack of robust verification requirements. In a similar vein, Altavilla et al. (2022) show that in Europe, despite being extended to small but creditworthy firms in sectors severely affected by the pandemic, guaranteed loans partially substituted for pre-existing debt, especially for riskier firms, thus shifting part of the existing credit risk from banks to governments.

[4] Similarly, the European Commission adopted a new crisis framework in March 2022 to support the economy in the context of Russia's invasion of Ukraine, whereby Member States can provide guarantees to ensure banks keep providing loans to companies affected by the current crisis (European Commission, 2022).

3 Relation to the literature

Our paper makes a number of contributions. First, our framework incorporates a bank's effort choice on the asset side in a model of financial fragility, where the probability of runs is endogenously determined. The paper therefore extends standard models of financial fragility (e.g., Goldstein and Pauzner, 2005, and Allen et al., 2018) to analyze the importance of the run threat for a bank's asset choice. This focus is in line with empirical evidence finding that banks are traditionally highly leveraged institutions, with debt being kept predominantly in the form of (both insured and uninsured) demandable (or short term) debt (e.g., Egan, Hortacsu and Matvos, 2017).

Another strand of literature has instead analyzed credit risk in the form of bank monitoring effort and the role of bank capital, but without including considerations of financial fragility. For example, Holmstrom and Tirole (1997) study the incentive problem for a bank to monitor a borrower and show how this incentive depends on the amount of capital the bank has. Hellmann, Murdock, and Stiglitz (2000), Repullo (2004), Morrison and White (2005), Dell'Ariccia and Marquez (2006), Allen, Carletti, and Marquez (2011), Mehran and Thakor (2011), and Dell'Ariccia, Laeven, and Marquez (2014) study settings where banks are subject to moral hazard in their monitoring decisions, and where equity capital helps improve bank incentives (see also Thakor, 2014, for a survey). It follows that banks may have incentives to raise capital even in the absence of capital requirements. None of these papers, however, studies how bank monitoring is affected by, and in turn affects, financial fragility in the form of bank runs. An exception is Kashyap, Tsomocos, and Vardoulakis (2019), who focus on the effect of capital and liquidity for credit and run risk. Instead, we are interested in the effects of loan guarantees for bank monitoring choice and the likelihood of runs.

Second, we contribute to the literature on the role of public loan guarantees by building a framework where guarantees introduced in response to crises impact the feedback between the asset and liability side of banks' balance sheets. As remarked above, guarantees on lending contracts are common in practice. Focusing on third-party loan guarantees for residential mortgages, Ahnert and Kuncl (2021) present a model where this type of guarantee decreases lending standards but improves market liquidity. In their model, lenders can pass default risk to an outside guarantor upon origination, thus avoiding costly screening. We also analyze loan guarantees upon origination, but in a context where these are not an alternative to bank screening.

Third, our paper is related to the literature studying alternative ways to transfer credit risk onto third parties after loan origination. For example, Parlour and Winton (2013) study the effects of credit default swaps (CDSs) on banks' monitoring incentives as an alternative to loan sales in secondary markets. They show that CDSs tend to dominate loan sales only for riskier credits, while their effects on bank monitoring depend on credit quality. In contrast, we focus on loan guarantees where banks retain both cash flow and control rights, and show that in the presence of these guarantees bank incentives depend on the level of capital, the size of the guarantee, and the nature of bankruptcy costs.

Fourth, a large strand of literature has focused on the role of government guarantees such as deposit insurance or other forms of implicit guarantees on banks' liabilities. On the one hand, these guarantees are thought to have a positive role in preventing panics among investors and help stabilize the financial system (e.g., Diamond and Dybvig, 1983). On the other hand, they may distort banks' incentives, leading to an increase in financial fragility (see, e.g., Calomiris, 1990, and Acharya and Mora, 2015). Reconciling the two views, more recent studies show that government guarantees can improve welfare because they induce banks to improve liquidity provision (Keister, 2016), although they may also increase the likelihood of runs (Allen et al., 2018). The idea that a government guarantee on deposits can actually be good for incentives has been studied in Cordella, Dell'Ariccia, and Marquez (2018), who show that, by reducing a bank's cost of funding, a deposit guarantee increases the return to the bank and creates greater incentives to monitor. In this paper, we focus on PGSs for loans rather than deposits and study how they affect bank behavior and financial stability through their interaction on the asset side of the balance sheet.

Finally, our analysis of the effect of loan guarantees on banks' incentives to engage in evergreening connects to a recent literature on zombie lending, i.e., the provision of credit to firms already in distress.[5] In Hu and Varas (2021), evergreening emerges from the existence of dynamic lending relationships and the advantages that a relationship bank can obtain from helping its borrowers to have a strong reputation. Bruche and Llobet (2014) show that zombie lending arises from limited liability. Relative to these papers, we focus on the effect that the introduction of a loan guarantee has on bank incentives to continue providing credit to firms in distress and highlight the role of bank capital. Related to this last point, Blattner et al. (2021) show empirically that, following the introduction of more stringent capital requirements in Portugal, weak banks started to provide credit to distressed firms for which the bank had been underreporting loan loss provisions prior the regulatory change. A similar result is also found in Schivardi, Sette, and Tabellini (2021), who show that during the 2008 financial crisis undercapitalized banks were more likely to provide credit to zombie firms than better capitalized ones. In line with this, in our framework poorly capitalized

[5] A number of earlier contributions focused on the Japanese experience; see, e.g., Peek and Rosengren (2005); and Caballero, Hoshi and Kashyap (2008).

banks have the greatest incentive to engage in evergreening.

4 The model

Consider a three date economy ($t = 0, 1, 2$) with banks and a large number of both firms and (atomistic) risk-neutral investors, with unitary endowment at date 0. On the asset side, each firm has a unit demand for a bank loan to finance a long term risky project[6], which, if held to maturity, yields a return $\widetilde{P}$, with

$$\widetilde{P} = \begin{cases} R\theta & \text{w.p. } q \\ 0 & \text{w.p. } 1-q \end{cases} .$$

The date 2 project return depends on the fundamentals of the economy θ and on the bank's effort choice $q \in [0, 1]$. The former captures the level of macroeconomic risk, while the latter represents the (endogenous) effort undertaken by a bank, which we will refer as either "underwriting" or "monitoring" throughout. We assume that the fundamentals of the economy θ are drawn from a uniform distribution in the range $[0, 1]$ with probability α; with complementary probability $1-\alpha$, θ is instead drawn from a uniform distribution in the range $[1, 2]$. In this respect, we interpret changes in α as shocks to the economy's fundamentals. The assumption $\theta \in [0, 2]$ guarantees that intermediation is feasible for any level of capital k. Furthermore, it captures the realistic case where the borrower defaults only in some states (i.e., when $\theta < 1$) and, in turn, as we show below, the loan guarantee is only paid when such default occurs.

Exerting greater underwriting effort q is costly and we assume that the bank bears a private cost of $c\frac{q^2}{2}$. For simplicity, we normalize the interest rate a bank receives on its loan to R, so that the bank's payoff is that of a standard debt contract. As shown in Figure 1, the bank receives full repayment for $\theta \geq 1$, while there is partial default for $\theta < 1$, with the bank receiving $R\theta$ and suffering losses $R(1-\theta)$.

Insert Figure 1

The loan can be liquidated early at $t = 1$, in which case it yields an amount whose value depends on the fundamental θ. Specifically, the liquidation value is equal to $L < 1$ for $\theta \in [0, \widehat{\theta})$ and to 1 for $\theta \in [\widehat{\theta}, 2]$. The idea is that the firm's project can only be liquidated at a cost when its returns are insufficient to fully repay the bank, while the asset's value upon liquidation is not impaired

[6] This specification implies that investors inelastically supply funds to the bank, and firms have an inelastic demand for loans, so that we can abstract from quantity effects on either the loan or the deposit market. On the liability side, investors need only have their reservation utility satisfied to be willing to deposit, consistent with the idea of a monopolistic deposit market. As specified below, the loan market is more stylized and the loan rate is set exogenously.

when its returns are high enough to fully repay the bank's loan.[7] The cutoff value $\widehat{\theta}$ is assumed to be close to but strictly below 1. To this end, we set $\widehat{\theta} \leq 1 - 2\varepsilon$ and, in most of the analysis, ε is taken to be arbitrarily close to 0. Finally, we assume that $\alpha \int_0^1 qR\theta d\theta + (1-\alpha)\int_1^2 qRd\theta - c\frac{q^2}{2} > 1$ for some q, so that granting loans to finance firms' projects dominates storing as long as the bank exerts a sufficiently high monitoring effort.

Each bank has (internal) capital of k and, at date 0, raises the remainder $1-k$ from investors in the form of demandable debt. The mass $1-k$ of investors at each bank holds a standard demandable deposit contract giving them the possibility to withdraw early or wait until the final date. At date 1, a depositor, whose outside option is normalized to 1, can redeem his deposit from the bank at par, i.e., for the same amount that was originally deposited, while he receives $r_2 > 1$ at date 2 if he waits until then.[8]

The promised repayments $\{1, r_2\}$ are paid as long as the bank has enough resources. If depositors choose to withdraw at date 1, the bank liquidates as much of its assets as needed to satisfy withdrawals and carries any remaining amount to date 2. If the bank has insufficient resources to meet depositors' demands at date 1, all its assets are liquidated and the $1-k$ depositors receive a pro-rata share of the liquidation value. By contrast, if the bank fails to repay depositors r_2 at date 2, the bank enters a bankruptcy procedure and depositors experience losses as a result.[9] For simplicity, we assume full bankruptcy costs, so that depositors receive nothing upon insolvency of the bank at date 2. The bankruptcy costs may originate either from coordination failures among the bank's creditors which makes it difficult and costly for them to seize the remaining value of the bank, or from the illiquidity of the bank's assets, where some value is lost when selling to alternative users/lenders. The different possible sources of bankruptcy costs will play an important role in the analysis of the loan guarantee scheme, as we discuss in detail below.[10]

[7] The assumption concerning the liquidation value L resembles the technical assumption made in Goldstein and Pauzner (2005) where there is no cost associated with early liquidation for high enough levels of the fundamental θ.

[8] While we assume that the bank offers demandable debt, Carletti, Leonello, and Marquez (2022) show, in a similar framework, that profit-maximing banks find it optimal to offer demandable deposit contracts without penalties for early withdrawals even at the risk of triggering a bank run. Hence, assuming $r_1 = 1$ is just a normalization in our context. Alternative justifications for the optimality of demandable debt relate to the presence of asymmetric information problems in credit markets (see, e.g., Flannery, 1986; and Diamond, 1991), conflicts between bank managers and debtholders (see e.g., Calomiris and Kahn, 1991; Diamond and Rajan, 2000, 2001; and Eisenbach, 2017), and idiosyncratic liquidity shocks to banks' depositors (e.g., Diamond and Dybvig, 1983).

[9] Considerable empirical evidence shows that bank bankruptcy costs are substantial. For example, James (1991) finds that when banks are liquidated, bankruptcy costs are 30 cents on the dollar.

[10] The asymmetric treatment of bankruptcy costs at date 1 and 2 is consistent with the idea that at least part of the cost associated with bankruptcy may stem from uncertainty related to the asset return. Importantly, this assumption does not qualitatively affect our results, as we show in Appendix B, where we replicate the analysis with a symmetric treatment of bankruptcy costs. Specifically, we consider the presence of bankruptcy costs at both date 1 and 2 as well as their absence at either date.

The state of the economy θ is realized at the beginning of date 1, but is publicly revealed only at date 2. After θ is realized at date 1, each depositor receives a private signal s_i of the form

$$s_i = \theta + \varepsilon_i, \tag{1}$$

where ε_i are small error terms that are independently and uniformly distributed over $[-\varepsilon, +\varepsilon]$. After the signal is realized, depositors decide whether to withdraw at date 1 or wait until date 2.

The timing of the model is as follows. At date 0, banks raise deposits with a deposit contract $\{1, r_2\}$, and then choose their monitoring effort q. At date 1, after receiving the private signal about the state of the fundamentals θ, depositors decide whether to withdraw early or wait until date 2. At date 2, the bank's portfolio return is realized and depositors that chose to wait are repaid.

5 Economy without guarantees

In this section, we characterize the allocation for the baseline case where there are no guarantees. We start by analyzing depositors' withdrawal decisions at date 1, taking the deposit contract $\{1, r_2\}$ and the riskiness of the portfolio q as given. Then, we move on to the choice of the monitoring effort q and the terms of the deposit contract r_2.

5.1 Depositors' withdrawal decision

Depositors base their withdrawal decisions on the signal they receive, as this gives them information about the economy's fundamentals θ and allows them to draw inferences on the actions of all other depositors at the bank. When he receives a high signal, a depositor expects the return of the bank's loan portfolio to be high and, at the same time, he expects that other depositors have also received a high signal. This lowers his incentives to withdraw early (i.e., run). Conversely, when a depositor receives a low signal, he expects a low return for the bank, and hence less cash available to repay depositors, and also a large number of depositors to run. As a result, he has a higher incentive to run. This suggests that depositors withdraw at date 1 when the signal is low enough, and wait until date 2 when the signal is sufficiently high.

To show this formally, we first examine two regions of extremely bad and extremely good fundamentals, where each depositor's action is based on the realization of the fundamentals θ irrespective of his beliefs about other depositors' behavior. We start with the lower region.

Lower Dominance Region. The lower dominance region of θ corresponds to the range $[0, \underline{\theta})$ in which running is a dominant strategy. Upon receiving a signal that suggests θ is in this region, a depositor is certain that the date 2 expected repayment is lower than the payment from withdrawing at date 1, even if no other depositors were to withdraw. Given the presence of bankruptcy costs, the depositor knows that at date 2 he will receive either $qr_2 > 1$ if the bank is solvent or 0 otherwise.[11] Thus, he has an incentive to run whenever the bank is insolvent, i.e., for θ below the threshold $\underline{\theta}(k)$, which is the solution to

$$R\underline{\theta} = (1-k)\, r_2. \tag{2}$$

Upper Dominance Region. The upper dominance region of θ corresponds to the range $[\overline{\theta}, 2]$ in which fundamentals are sufficiently good that waiting to withdraw at date 2 is a dominant strategy. The higher liquidation value for $\theta \geq \widehat{\theta}$, together with the promised date 1 repayment of 1, guarantees that the bank liquidates only 1 unit of its investment for each withdrawing depositor, thus preventing strategic complementarity in depositors' decisions. Given that $\widehat{\theta} > \underline{\theta}$, the bank's resources are enough to fully repay depositors' promised repayment at date 2. This implies that, for any $\theta \geq \widehat{\theta}$, a depositor waiting until date 2 expects to receive $qr_2 > 1$. It follows that $\overline{\theta} = \widehat{\theta}$, so that the upper dominance region corresponds to the region where there is no impairment in the liquidation value of the assets, as described above.[12]

The Intermediate Region. When the signal indicates that θ is in the intermediate range, $[\underline{\theta}, \overline{\theta})$, a depositor's decision to withdraw early depends on the realization of θ as well as on his beliefs regarding other depositors' actions. To see how, we first calculate a depositor's utility differential between withdrawing at date 2 and at date 1. Using n to represent the fraction of depositors who choose to withdraw early, this differential is given by

$$v(\theta, n) = \begin{cases} qr_2 - 1 & \text{if } 0 \leq n \leq \widehat{n}(\theta) \\ 0 - 1 & \text{if } \widehat{n}(\theta) \leq n \leq \overline{n} \\ 0 - \frac{L}{(1-k)n} & \text{if } \overline{n} \leq n \leq 1 \end{cases}, \tag{3}$$

where $\widehat{n}(\theta)$ solves

$$R\theta \left(1 - \frac{\widehat{n}(1-k)}{L}\right) - (1-\widehat{n})(1-k)\, r_2 = 0, \tag{4}$$

while $\overline{n}$ solves

$$L = \overline{n}(1-k).$$

[11] The condition $qr_2 > 1$ is required for investors to deposit and for intermediation to be feasible. If $qr_2 < 1$, depositors would never find it optimal to wait until date 2 and would strictly prefer to withdraw early, at date 1. Anticipating this, all depositors would prefer to pursue whatever alternative investment is available to them yielding 1 rather than deposit at the bank. Hence, a minimum requirement for intermediation to be feasible is that the bank chooses a high enough level of monitoring so that, given the equilibrium r_2, $qr_2 > 1$. This can readily be achieved for c sufficiently low and/or R sufficiently high.

[12] As $\widehat{\theta} \leq 1$, the discontinuity in the distribution of θ due to the parameter α at $\theta = 1$ is included in the upper dominance region and thus does not affect the characterization of the panic threshold θ^* below.

The threshold $\widehat{n}(\theta)$ represents the proportion of depositors running at which a bank is no longer able to repay r_2 to those waiting until date 2, while $\overline{n}$ captures the number of withdrawing depositors at which a bank liquidates the entire portfolio at date 1. As illustrated in Figure 2, when $1-k \leq L$ the function $v(\theta, n)$ is constant in n and is equal to $qr_2 - 1 > 0$ if $\theta \geq \underline{\theta}$ and to -1 if $\theta < \underline{\theta}$. Hence, in this case $v(\theta, n)$ is either positive or negative depending on whether θ is above or below $\underline{\theta}$. This implies that a depositor's incentive to run is independent of what others do or, in other words, that runs are only triggered by the fear that fundamentals are low.

Insert Figure 2

Figure 3 illustrates the case when $1 - k > L$ and shows that the function $v(\theta, n)$ is constant and positive for $0 \leq n \leq \widehat{n}(\theta)$, while it is always below zero in the range $\widehat{n}(\theta) \leq n \leq \overline{n}$.

Insert Figure 3

Since $1 - k > L$ and each depositor is promised 1 unit at date 1, the bank has to liquidate more units of the project than the number of withdrawing depositors, thus being forced to liquidate all its assets prematurely if many depositors demand their funds at date 1. This introduces strategic complementarities in depositors' withdrawal decisions, as is typical in models of runs (e.g., Goldstein and Pauzner, 2005): the expected payoff of depositors waiting until date 2 is decreasing in the proportion n of depositors withdrawing at date 1, so that their incentive to run increases with n. Hence, a depositor's withdrawal decision depends on other depositors' behavior and runs are driven by fears of large withdrawals in the form of panics.

Throughout, we focus our results on the limiting case where $\varepsilon \rightarrow 0$, so that the noise in depositors' information becomes vanishingly small. This implies that all depositors behave alike: they all either withdraw at date 1, or wait until date 2. The following proposition characterizes depositors' withdrawal decisions.

Proposition 1 *The model has a unique equilibrium for depositors' withdrawal decisions, where depositors only withdraw below a certain threshold of fundamentals, as follows:*

a) When $1 - k \leq L$, only fundamental runs occur for θ below the threshold $\underline{\theta}(k)$, where

$$\underline{\theta}(k) = \frac{(1-k)\, r_2}{R}, \tag{5}$$

with $\underline{\theta}(k)$ decreasing in k: $\frac{\partial \underline{\theta}(k)}{\partial k} < 0$.

b) When $1-k>L$, panic runs also occur for θ below the threshold $\theta^*(q,k,L,r_2)$, where

$$\theta^*(q,k,L,r_2)=\underline{\theta}\frac{qr_2-\pi_1}{qr_2-\pi_1\frac{(1-k)}{L}}, \tag{6}$$

and $\pi_1=\int_0^{\overline{n}}dn+\int_{\overline{n}}^1\frac{L}{(1-k)n}dn$. The threshold $\theta^*(q,k,L,r_2)\in(\underline{\theta}(k),1)$ decreases with q, L, and k: $\frac{\partial\theta^*(q,k,L,r_2)}{\partial q}<0$, $\frac{\partial\theta^*(q,k,L,r_2)}{\partial L}<0$, and $\frac{\partial\theta^*(q,k,L,r_2)}{\partial k}<0$.

The proposition shows the importance of bank capital for run risk. When a bank is well capitalized (i.e., when $1-k\leq L$), runs are driven only by poor fundamentals, and the critical threshold $\underline{\theta}$ is decreasing in the amount of capital k. In contrast, when a bank has little capital (i.e., when $1-k>L$), it is exposed to runs over a larger range of fundamentals (i.e., for $\theta<\theta^*$ with $\theta^*>\underline{\theta}$) due to the presence of strategic complementarities. The panic threshold θ^* decreases with the monitoring effort q, the level of capital k, and the liquidation value L. A higher q increases depositors' expected payoff from waiting until date 2, while a higher k or a higher L reduces the bank's liquidation needs, thus mitigating strategic complementaries. Thus, banks with little capital face higher run risk, and we assume that at the limit $\theta^*\to\overline{\theta}$ as $k\to 0$.[13]

The role of capital highlighted in Proposition 1 is, to our knowledge, novel, and raises the question of which type of run may be more relevant in practice. As discussed in the survey by Goldstein (2012), there is a strong link between crises and fundamentals, with coordination failures amplifying depositors' response to fundamentals. Our model links this discussion to the level of bank capital. In normal times, banks tend to be well capitalized on average, and thus we would expect fundamental crises to be the most relevant cases. By contrast, in a downturn banks tend to have more stretched levels of capital, and thus crises may also arise due to coordination failures among depositors. As we show below, however, for the most part this distinction does not matter much for understanding the effects of a loan guarantee on financial stability.

5.2 Bank's date 0 decisions

Having characterized depositors' withdrawal decisions, we now solve for banks' underwriting standards q and the repayment r_2. We use θ^R to denote the relevant run threshold, i.e., $\theta^R=\underline{\theta}$ when $1-k\leq L$ and $\theta^R=\theta^*$ when $1-k>L$.

Each bank chooses q and r_2 anticipating depositors' withdrawal decisions at date 1, thus solving

[13] Letting $k\to 0$ in the expression for θ^* in (6), it can be seen that θ^* approaches its maximum, i.e., $\theta^*\to\overline{\theta}$ as L decreases, as this leads to an increase in the strategic complementarity among depositors' withdrawal decisions.

the following problem:

$$\max_{q,r_2} \Pi = \alpha \int_0^{\theta^R} q \max\left\{R\theta\left(1-\frac{1-k}{L}\right),0\right\} d\theta + \alpha\int_{\theta^R}^1 q\left[R\theta-(1-k)\,r_2\right] d\theta$$
$$+ (1-\alpha)\int_1^2 q\left[R-(1-k)\,r_2\right] d\theta - \frac{cq^2}{2} \quad (7)$$

subject to

$$\underbrace{\alpha\int_0^{\theta^R} \min\left\{\frac{L}{1-k},1\right\} d\theta}_{\text{utility obtained in a run}} + \underbrace{\alpha\int_{\theta^R}^1 qr_2 d\theta + (1-\alpha)\int_1^2 qr_2 d\theta}_{\text{utility obtained if no runs occur}} \geq \underbrace{1}_{\text{outside option}} \quad (8)$$

and

$$\Pi \geq k. \quad (9)$$

The first three terms in (7) capture the three instances when the bank accrues positive profits at date 2, while the last term captures the monitoring cost. When a run occurs, a bank with $1-k \leq L$ does not liquidate its entire portfolio at date 1 and thus obtains the return $R\theta$ on the $1-\frac{1-k}{L}$ remaining units of assets at date 2 with probability q. When no run occurs, a bank makes positive profits at date 2 as given by the project return ($R\theta$ for $\theta \in (\theta^R, 1)$ and R for $\theta \in [1,2]$) minus the repayment $(1-k)\,r_2$ to depositors.

The condition in (8) represents depositors' participation constraint and requires that the expected promised repayment from depositing be no lower than depositors' outside option. The expected repayment is given by the minimum between the pro-rata share $\frac{L}{1-k}$ and the promised repayment 1 if there is a run (i.e., when $\theta \leq \theta^R$) and qr_2 if there is no run (i.e., $\theta > \theta^R$). Finally, the inequality in (9) is simply a non-negativity constraint on bank expected profits. We have the following result.

Proposition 2 *The equilibrium is as follows:*

a) When $1-k \leq L$, each bank chooses $\underline{q}$ as a solution to

$$\alpha\int_0^{\underline{\theta}} R\theta\left(1-\frac{(1-k)}{L}\right) d\theta + \alpha\int_{\underline{\theta}}^1 \left[R\theta-(1-k)\,r_2\right] d\theta + (1-\alpha)\int_1^2 \left[R-(1-k)\,r_2\right] d\theta - cq = 0, \quad (10)$$

where $r_2 > 1$ solves (8) holding with equality;

b) When $1-k > L$, each bank chooses q^* as a solution to

$$\alpha\int_{\theta^*}^1 \left[R\theta-(1-k)\,r_2\right] d\theta + (1-\alpha)\int_1^2 \left[R-(1-k)\,r_2\right] d\theta - \alpha\frac{\partial\theta^*}{\partial q} q\left[R\theta^*-(1-k)\,r_2\right] - cq = 0, \quad (11)$$

where $r_2 > 1$ solves

$$-\alpha\frac{\partial\theta^*}{\partial r_2}\left[R\theta^* - (1-k)\, r_2\right] - \alpha\int_{\theta^*}^{1}(1-k)\, d\theta - (1-\alpha)\int_{1}^{2}(1-k)\, d\theta = 0 \tag{12}$$

when $\mu = 0$, and is the lowest r_2 solving (8) holding with equality when $\mu > 0$, where μ is the Lagrange multiplier on depositors' participation constraint as defined in Appendix A.

In choosing q, a bank trades off the marginal cost cq of an increase in q with its marginal benefit as captured by the first three terms in either (10) or (11). A higher q increases the expected profit when there is no run for $\theta > \theta^R$ for any k and also the profits when $1-k \leq L$ in the event of a run for $\theta < \underline{\theta}$ in (10). In addition, when $1-k > L$, a higher q reduces depositors the probability of a depositor run as given by $\frac{\partial\theta^*}{\partial q} < 0$ in (11).

Proposition 2 shows that the determination of r_2 also depends on the level of capital of the bank. Banks with $1-k \leq L$ choose the lowest possible repayment r_2 consistent with depositors being willing to provide funds to the bank. By contrast, banks with $1-k > L$ also account for the potentially beneficial effect that a higher r_2 has on the run threshold θ^* since θ^* is decreasing in r_2, at least for some values of r_2. As a result, a bank may find it optimal to choose a repayment r_2 which leaves depositors' participation constraint (8) slack.

6 Public loan guarantee schemes

So far, we have characterized the equilibrium – depositors' withdrawal decisions and bank underwriting choices – for a given α, under the assumption that the project has a positive NPV and banks are willing to lend. Now we consider the case of a negative shock through an increase in α, which we interpret as a "crisis" episode leading to a worsening of the distribution of fundamentals θ and consequently to a reduction of the return banks obtain. It follows that for a large enough shock (i.e., a large enough increase in α), banks may no longer find it optimal to grant loans, at least not without having to raise the interest rate. This calls for support measures such as loan guarantees that offer credit protection against low realization of the fundamentals, thus effectively providing banks with a subsidy tied to their lending activities.

In this section, we study how the introduction of loan guarantees affect bank lending through their effects on bank underwriting standards and investors' behavior. To this end, we consider that the guarantee is introduced after the shock, as captured by an increase in α, say from α_0 to α_1, which occurs unexpectedly after the bank has secured funding, but before the fundamental of the economy θ is realized. In this respect, we consider a situation such as the Covid pandemic, where loan guarantees were extended in the face of an unexpected crisis that is still unfolding. Later, in Section 8.1, we discuss the case when the introduction of the guarantees may also affect bank funding costs. Similarly, since borrowers have unit demand for loans, we assume that there is no pass-through of the guarantees to the loan rate.

We consider a loan guarantee, denoted as a first-loss guarantee, where losses are first attributed to the government up to a certain limit, and only then to the credit intermediary.[14] In other words, the government guarantees any loss occurring at date 2 up to an amount Rx, with any remaining losses being borne by the bank. Formally, the government will transfer an amount $R \min\{x, 1-\theta\}$ to the bank when the borrower is unable to repay the promised amount R, with

$$x < \frac{1-k}{R}. \tag{13}$$

This assumption ensures that the government transfer alone is not sufficient to fully shield depositors from losses, thus preserving depositors' incentives to run. Thus, as illustrated in Figure 4, the bank now obtains the full repayment R for $\theta \in [1-x, 1]$ and a greater payoff $R(\theta+x) < R$ for $\theta \in [0, 1-x)$ where the losses are greater than the guarantee provided.

Insert Figure 4

Within this scheme, we consider two cases concerning the treatment of the loan guarantee in case the bank is insolvent at date 2. In the first case, denoted as "full bankruptcy costs," the amount provided by the government is dissipated in bankruptcy in the same way as the return of any other asset. In the second case, denoted as "bankruptcy protected," the transfer Rx from the government to a bank is instead protected from bankruptcy costs and can thus be used to repay investors. The first case captures the idea that the bankruptcy costs primarily originate from inefficiencies in bankruptcy procedures due to hold-up problems among creditors or inefficient judicial systems and, as a result, resources are lost if the bank defaults at date 2. The second case would be consistent with a setting where bankruptcy costs primarily stem from illiquidity associated with selling assets. The guarantee paid by the government would likely be in cash or other such liquid assets and less subject to dissipation. Importantly, assumption (13) implies that the bank can only profit from the guarantee at date 2 in states of the world when its monitoring decisions have paid off (i.e., with probability q). For both of these cases, we assume that the interest rate on

[14] While the specific terms may differ across countries, PGSs take essentially one of two forms: first-loss or loss-sharing (see, for example, European Commission, 2020, for the two schemes used in Europe during the pandemic). In the latter, losses are sustained proportionally by the credit institutions and the state in some pre-determined proportions. We show in Section 6.3 that the main insights of the analysis carry over to the loss-sharing scheme.

the loan, R, remains unchanged, implying that there is no pass through of the guarantee Rx onto loan rates. This is consistent with the assumption of unit demand for loans by borrowers.

6.1 First-loss guarantee scheme with full bankruptcy costs

As in the baseline case, we start by characterizing depositors' withdrawal decisions and then move on to the choice of q by banks. We use the subscript x to indicate the case of the first-loss guarantee of size x with full bankruptcy costs.

Proposition 3 *With a first-loss guarantee x and full bankruptcy costs, runs occur for $\theta < \theta_x^R < \theta^R$ as given by*

$$\theta_x^R = \theta^R - x, \tag{14}$$

where $\theta_x^R = \underline{\theta}_x$ and $\theta^R = \underline{\theta}$ for $1-k \leq L$, while $\theta_x^R = \theta_x^*$ and $\theta^R = \theta^*$ for $1-k > L$. The threshold θ_x^R decreases with x: $\frac{\partial \theta_x^R}{\partial x} = -1 < 0$.

The introduction of loan guarantees reduces the run thresholds for any given level of bank capital. A higher x increases the range in which the bank is able to make the promised repayment to depositors at date 2, thus reducing their incentives to withdraw prematurely. The threshold θ_x^R depends linearly on the amount x because this accrues to depositors only if the bank's monitoring is successful and the bank is solvent.

Anticipating depositors' withdrawal behavior, each bank solves the following optimization problem:

$$\max_q \alpha \int_0^{\theta_x^R} q \max\left\{R(\theta+x)\left(1-\frac{(1-k)}{L}\right), 0\right\} d\theta + \alpha \int_{\theta_x^R}^{1-x} q\left[R(\theta+x)-(1-k)r_2\right] d\theta \tag{15}$$
$$+\alpha \int_{1-x}^{1} q\left[R-(1-k)r_2\right] d\theta + (1-\alpha)\int_1^2 q\left[R-(1-k)r_2\right] d\theta - \frac{cq^2}{2},$$

where r_2 is characterized in Proposition 2 since the guarantee scheme is assumed to be unanticipated, and θ_x^R denotes the relevant run threshold, i.e., $\theta_x^R = \underline{\theta}_x$ when $1-k \leq L$ and $\theta_x^R = \theta_x^*$ when $1-k > L$. The terms in (15) are similar to those in (7) in the baseline model, with two main differences. First, as indicated in the first two terms, the bank now obtains a per-unit return $R(\theta+x)$ at date 2 instead of $R\theta$ whenever the loan is not fully repaid. Second, as shown in the third term, the bank is able to obtain the full repayment R in the larger range of values of $\theta \in [1-x, 2]$ rather than for $\theta \in [1, 2]$. Note that in (15), for ease of notation, we continue to express bank profits as a function of a generic α, but recognizing that we have in mind a case where a negative shock has occurred so that this weight has increased.

Each bank chooses the underwriting effort $\underline{q}_x$ as a solution to

$$\alpha \int_0^{\underline{\theta}_x} R(\theta + x)\left(1 - \frac{(1-k)}{L}\right) d\theta + \alpha \int_{\underline{\theta}_x}^{1-x} [R(\theta + x) - (1-k) r_2]\, d\theta + \alpha \int_{1-x}^{1} [R - (1-k) r_2]\, d\theta + (1-\alpha) \int_1^2 [R - (1-k) r_2]\, d\theta - cq = 0 \tag{16}$$

when $1 - k \leq L$, and q_x^* when $1 - k > L$ as a solution to

$$\alpha \int_{\theta_x^*}^{1-x} [R(\theta + x) - (1-k) r_2]\, d\theta + \alpha \int_{1-x}^{1} [R - (1-k) r_2]\, d\theta + (1-\alpha) \int_1^2 [R - (1-k) r_2]\, d\theta - \alpha \frac{\partial \theta_x^*}{\partial q} q [R(\theta_x^* + x) - (1-k) r_2] - cq = 0. \tag{17}$$

The interpretation of the various terms in (16) and (17) is the same as for the terms in (10) and (11).

In the following proposition, we characterize the effect that the guarantees have on banks' underwriting effort decisions. We use q_x^R to denote either $\underline{q}_x$ or q_x^*, depending on the level of bank capital.

Proposition 4 *For any given level of k, the introduction of a first-loss loan guarantee x with full bankruptcy costs increases bank underwriting effort:* $\frac{dq_x^R}{dx} > 0$.

This proposition highlights that the introduction of the loan guarantee induces banks to reduce the riskiness of their portfolios through improved underwriting incentives for all banks, irrespective of how much capital they have. The mechanism resembles a "charter value" (e.g., Keeley, 1990) in that the bank has more to lose when it fails. In fact, the loan guarantee increases the bank's expected profits both through an increase in the probability of survival until date 2 (i.e., a reduction of the threshold θ_x^R) and an increased per-unit return in case of survival. Given this, the bank has stronger incentives to remain active until the final date, which can be achieved through a higher underwriting effort.

It is worthwhile noting that the unambiguously positive effect of the loan guarantee on bank underwriting effort obtains for $x < \frac{1-k}{R}$. With full bankruptcy costs, this condition, which ensures that the guarantee is insufficient to fully cover the promised repayment to depositors, implies that the bank benefits from the guarantee only when its project succeeds. If the guarantee was such that $x \geq \frac{1-k}{R}$, depositors would no longer impose any discipline on the bank through the threat of a run and the bank would receive a portion of the loan guarantee even when its project fails, with probability $1 - q$. At the margin, this payment to the bank would reduce its monitoring incentives.

The results in this section help provide guidance on policy initiatives supporting access to credit through loan guarantees, as discussed in Section 2. To the extent that many of these policy interventions entail partial guarantees, Propositions 3 and 4 suggest that the concerns regarding possible moral hazard may be overstated, and provide a channel for financial stability to actually improve as a result. Our results also highlight the importance of the design of any such policy initiatives, which should ensure that any benefit a lender receives directly from the guarantee be primarily obtained when it is sufficiently diligent in monitoring its lending activities and properly underwriting loans, as is the case studied here for sufficiently small guarantees.

6.2 Bankruptcy-protected first-loss guarantee scheme

In this section, we consider the possibility that the government's transfer x is sheltered from other frictions which lead to losses that result from bankruptcy. Specifically, we assume that, in case of default by the bank, depositors receive these amounts even if any revenues stemming from the bank's loans are lost in bankruptcy. This would be consistent with a setting where bankruptcy costs primarily stem from illiquidity associated with selling assets, be they loans or otherwise. The guarantee paid by the government would likely be in cash or other such liquid assets and less subject to dissipation. As before we consider the case where $x < \frac{1-k}{R}$. The following proposition characterizes depositors' withdrawal decisions.

Proposition 5 *With a bankruptcy-protected guarantee x, the run risk is as follows:*

a) When $1-k \leq L$, runs occur for $\theta < \underline{\theta}_x^P = \underline{\theta}_x < \underline{\theta}$ as given by (14);

b) When $1-k > L$, runs occur for $\theta < \theta_x^{*P}$, with $\theta_x^{*P} < \theta_x^* < \theta^*$ as given by the solution to

$$\pi_1 = \int_0^{\widehat{n}_x(\theta)} q r_2 dn + \int_{\widehat{n}_x(\theta)}^{\overline{n}} q \frac{Rx\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)} dn + \int_0^{\overline{n}} (1-q) \frac{Rx\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)} dn, \tag{18}$$

where π_1 is as in Proposition 1. The run threshold θ_x^{*P} decreases with q, k, and x: $\frac{\partial \theta_x^{*P}}{\partial q} < 0$, $\frac{\partial \theta_x^{*P}}{\partial k} < 0$, and $\frac{\partial \theta_x^{*P}}{\partial x} < -1$.

As for the case with full bankruptcy costs, the introduction of the guarantee induces a reduction of the run probability. When $1-k \leq L$, the run threshold is the same as for the case of full bankruptcy costs because what depositors obtain when the bank is insolvent, $\frac{Rx}{1 \quad k}$, is always lower than what they obtain when withdrawing. By contrast, when $1-k > L$, the loan guarantee is now more effective in reducing depositors' incentives to run relative to the case of full bankruptcy costs, so that $\theta_x^{*P} < \theta_x^*$. In the presence of strategic complementarities, depositors compare the expected payoff at date 1 with that from waiting until date 2. Both payoffs depend on other depositors' actions. Thus, depositors take into account the possibility that, depending on the size of n, they may obtain a pro-rata share both at date 1 or date 2, and that the guarantee x increases the payoff they obtain at date 2, as evident in the last two terms on the RHS in (18). This reinforces their incentives to wait until date 2.

As in the baseline case, the run threshold θ_x^{*P} is decreasing in both q and k, as well as in x. Given that the transfer is not lost in bankruptcy, when the bank is insolvent depositors expect to receive a pro-rata share of the bank's available resources $\frac{Rx\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}$, which is a function of the fraction n of withdrawing depositors. Importantly, the sensitivity of the run threshold θ_x^{*P} to the transfer x is now higher relative to the case with full bankruptcy costs and it depends on q, i.e., $\frac{\partial \theta_x^{*P}}{\partial x} < \frac{\partial \theta_x^{*}}{\partial x} = -1$. The intuition behind the greater sensitivity lies in the extra effect of the guarantee in terms of higher payoffs at date 2 whenever the bank is unable to repay the promised amount.

The bank's maximization problem for the choice of q is similar to the one characterized in (15), with the only difference being that the relevant run threshold is either $\underline{\theta}_x^P$ or θ_x^{*P} instead of θ_x^R. Each bank chooses underwriting effort $\underline{q}_x^P = \underline{q}_x$ as the solution to (16) when $1-k \leq L$, and q_x^{*P} as a solution to

$$\alpha \int_{\theta_x^{*P}}^{1-x} \left[R(\theta + x) - (1-k)\, r_2\right] d\theta + \alpha \int_{1-x}^{1} \left[R - (1-k)\, r_2\right] d\theta + (1-\alpha) \int_1^2 \left[R - (1-k)\, r_2\right] d\theta$$

$$-\alpha \frac{\partial \theta_x^{*P}}{\partial q} q \left[R\left(\theta_x^{*P} + x\right) - (1-k)\, r_2\right] - cq = 0 \tag{19}$$

when $1-k > L$.

Proposition 6 *The impact of a bankruptcy-remote first-loss guarantee x on bank monitoring effort is as follows:*

a) When $1-k \leq L$, the introduction of the loan guarantee leads to the same positive impact on $\underline{q}_x^P$ as in Proposition 4, i.e., $\frac{d\underline{q}_x^P}{dx} > 0$;

b) When $1-k > L$, there exists a value of k denoted as $\widehat{k}_x^P < 1-L$ such that introducing the loan guarantee reduces bank effort for $k < \widehat{k}_x^P$, but increases effort as $k \to 1-L$: $\frac{dq_x^{*P}}{dx} < 0$ for $k < \widehat{k}_x^P$ and $\frac{dq_x^{*P}}{dx} > 0$ for $k \to 1-L$.

The proposition shows that the introduction of the loan guarantee increases bank monitoring effort for banks with a sufficiently high level of capital, while it decreases it for banks with very little capital when they are subject to significant run risk. As in the case with full bankruptcy costs, the bank obtains nothing when its monitoring is unsuccessful (i.e., with probability $1-q$) since x is assumed to be relatively small, as defined in (13). However, differently from before, the transfer x accrues to depositors whenever the bank is insolvent. When $1-k \leq L$, the more favorable treatment of the guarantee under bankruptcy does not affect depositors' incentives so that, in turn, bank monitoring responds in the same way as before. By contrast, when $1-k > L$, the fact that depositors receive Rx with probability $1-q$ introduces a disincentive to monitor as a result of two effects: first, Rx reduces the sensitivity of the run threshold θ_x^{*P} to changes in q and, second, by reducing θ_x^{*P}, it reduces the losses associated with an increase in the probability of a run due to low monitoring effort by the bank. These two effects combined lead highly levered banks (i.e., those with $k < \widehat{k}_x^P < 1-L$) to exert less effort q. In other words, the potential negative impact of the loan guarantee on bank monitoring derives purely from its effect on depositor behavior and run risk, and in particular from the reduced sensitivity of the run threshold θ_x^{*P} to the bank's choice of q. For sufficiently poorly capitalized banks (i.e., $k < \widehat{k}_x^P$), these negative effects dominate.

6.3 Guarantee scheme with loss-sharing

In this section we analyze a second type of guarantee, which has also been used during the Covid pandemic and which we denote as "loss-sharing." We first show that the results obtained in Section 6 in the case of first-loss loan guarantees are qualitatively the same. We then compare the two schemes in terms of their effectiveness.

Suppose that the government commits to cover a fraction $y \in (0,1)$ of bank losses $R(1-\theta)$, so that the bank's per unit loan return is equal to $max\,\{R, R\theta + R\,(1-\theta)\,y\}$. We have the following result, which encompasses both the case of full bankruptcy costs and the one where transfers are bankruptcy-protected.

Proposition 7 *The introduction of a loss-sharing guarantee y leads to the following:*

a) In the case of full bankruptcy costs:

1. *Runs occur for $\theta < \theta_y^R \equiv \{\underline{\theta}_y, \theta_y^*\}$, where*

$$\theta_y^R = \frac{\theta^R - y}{1-y} \tag{20}$$

   and $\theta^R = \underline{\theta}$ when $1-k \leq L$ and $\theta^R = \theta^*$ when $1-k > L$ as characterized in Proposition 1.

2. *For any level of k, the bank's underwriting effort q_y^R increases in the guaranteed amount: $\frac{dq_y^R}{dy} > 0$.*

b) In the case where the government's transfers are protected from bankruptcy:

1. *Runs occur for* $\theta < \underline{\theta}_y^P = \underline{\theta}_y$ *when* $1-k \leq L$ *and for* $\theta < \theta_y^{*P}$ *when* $1-k > L$*, where* $\theta_y^{*P} > \underline{\theta}_y^P$ *solves*

$$\pi_1 = \int_0^{\widehat{n}_y(\theta)} q r_2 dn + \int_{\widehat{n}_y(\theta)}^{\overline{n}} q \frac{R(1-\theta)\,y\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn + \int_0^{\overline{n}} (1-q) \frac{Ry\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn,$$

2. *Bank effort* $\underline{q}_y$ *increases with the introduction of the guarantees when* $1-k \leq L$*:* $\frac{d\underline{q}_y}{dy} > 0$*. Moreover, there exists a value* $\widehat{k}_y^P \in (0, 1-L)$ *such that* q_y^* *decreases with the introduction of guarantees for any* $k < \widehat{k}_y^P$*:* $\frac{dq_y^*}{dy} < 0$ *for* $k < \widehat{k}_y^P$*.*

The scheme where the guarantee requires banks to share any losses on a proportional basis delivers the same qualitative results in terms of financial fragility and bank underwriting effort as the first-loss scheme: for any level of bank capital, the guarantee reduces the run threshold relative to the case with no guarantees. Also, as before, the effect of the loan guarantee on bank monitoring incentives depends on the treatment of the guarantee in bankruptcy and the level of capital, in that bank monitoring increases except for very poorly capitalized banks (i.e., with $k < \widehat{k}_y^P$) if they are exposed to significant run risk when the guarantee is protected from bankruptcy.

While the two guarantee schemes - first-loss or loss-sharing - deliver qualitatively similar results, a natural question that arises is whether one of them may be more effective or cost-efficient. To see this, we compare the two schemes (GS) under the maintained assumption that the guaranteed amount is lost in bankruptcy and, for tractability, we restrict attention to well-capitalized banks with $1-k \leq L$. We maintain the subscript x when referring to the first-loss guarantee (GS_x) and the subscript y to denote the loss-sharing scheme (GS_y).

To compare the two schemes, we consider the case where the sizes x and y of the guarantees are set, all things equal, to lead to the same run threshold: $\underline{\theta}_x = \underline{\theta}_y$. Equating these two, we specify $\underline{y}$ as the level of y for which the two guarantee schemes implement the same probability of a run. Hence, $\underline{y}$ solves

$$\frac{\underline{\theta} - y}{1-y} = \underline{\theta} - x,$$

and is equal to

$$\underline{y} = \frac{x}{1 - \max\{\underline{\theta} - x, 0\}} \geq x,$$

since $\underline{\theta} - x \equiv \underline{\theta}_x < 1$.

For a given size of the transfer x, the guarantee scheme GS_x entails a disbursement for the government equal to

$$GD_x = \alpha \int_0^{\underline{\theta}-x} Rx \left(1 - \frac{1-k}{L}\right) d\theta + \alpha \int_{\underline{\theta}-x}^{1-x} Rx d\theta + \alpha \int_{1-x}^{1} R(1-\theta)\, d\theta \tag{21}$$
$$= \alpha \left[Rx - \frac{Rx^2}{2} - Rx\left(\underline{\theta} - x\right) \frac{1-k}{L} \right],$$

while GS_y entails a disbursement equal to

$$GD_y = \alpha \int_0^{\underline{\theta}-x} R(1-\theta)\, y \left(1 - \frac{1-k}{L}\right) d\theta + \alpha \int_{\underline{\theta}-x}^{1} R(1-\theta)\, y \tag{22}$$
$$= \alpha \left[\frac{Ry}{2} - Ry\left(\underline{\theta} - x\right) \frac{1-k}{L} + \frac{Ry}{2} \frac{1-k}{L} \left(\underline{\theta} - x\right)^2 \right]$$

Comparing GD_x and GD_y when $y = \underline{y}$, we have the following result.

Proposition 8 *For any $x > 0$, when both schemes are designed to achieve the same run threshold, the first-loss guarantee scheme entails a larger disbursement for the government than the loss-sharing scheme, but it induces the bank to choose a higher q.*

The proposition shows that, while the first-loss guarantee scheme provides greater incentives to the bank through improved bank underwriting standards, it achieves this at a higher cost. Hence, neither type of scheme unambiguously dominates the other, suggesting that fine tuning the design of the guarantee scheme may not be as important as just getting one in place in the event of a crisis.

7 Inefficient liquidation and zombie lending

So far, we have characterized the effect of loan guarantees on bank risk-taking in terms of monitoring effort. In this section, we analyze another form of risk-taking. Specifically, we focus on banks' incentives to engage in "evergreening," or in other words inefficient loan continuation, and how these are affected by loan guarantees. To do so, we modify the model slightly and assume that at date 1 a bank can choose whether to liquidate its loan portfolio or continue until the final date. Such choice is made after depositors' withdrawal decisions and thus does not interfere with how depositors evaluate their private signals.

To isolate banks' evergreening incentives, we start by analyzing a bank's liquidation decision at date 1 in a setting where runs at $t = 1$ are not possible and there are no loan guarantees. In this case, each bank compares the expected return of the loan at date 2 with its liquidation value at date 1, net of depositors' repayments, and it chooses to liquidate if θ falls below the threshold θ_L^B as given by the solution to

$$L-(1-k)r_2=q\left(R\theta-(1-k)r_2\right),$$

which is equivalent to

$$\theta_L^B=\frac{L-(1-q)(1-k)r_2}{qR}. \tag{23}$$

A bank's liquidation decision may not be socially optimal. To see why, we compare it with that of a social planner who finds it optimal to liquidate the portfolio when θ falls below the threshold θ_L^{SP} as given by the solution to

$$L=q\theta R,$$

and is thus equal to

$$\theta_L^{SP}=\frac{L}{qR}. \tag{24}$$

We have the following result.

Lemma 1 *In an economy without runs, banks liquidate too little relative to what is socially optimal:* $\theta_L^B<\theta_L^{SP}$. *The difference* $\theta_L^{SP}-\theta_L^B$ *measures the extent of evergreening and is decreasing in* k*:* $\frac{\partial(\theta_L^{SP}-\theta_L^B)}{\partial k}<0$.

We now go back to the case where depositors can withdraw at $t=1$. This implies that loans can be liquidated at date 1 for two reasons: either because a run occurs, or because a bank prefers to liquidate its portfolio prematurely even if no run occurs. To see when either case is relevant, we compare banks' liquidation threshold θ_L^B with the run threshold $\theta^R=\{\underline{\theta},\theta^*\}$ as characterized in Section 5**.**

Lemma 2 *The comparison between* θ_L^B *and* θ^R *depends on the level of bank capital* k. *Let* $\overline{k}_L=1-\frac{L}{r_2}>1-L$. *Then,* $\theta_L^B\leq\theta^R$ *for* $k\leq\overline{k}_L$ *and* $\theta_L^B>\theta^R$ *otherwise.*

When a bank has little capital and is exposed to panic runs, it never finds it optimal to liquidate its portfolio at date 1 when a run does not occur. In other words, given $\theta^*>\theta_L^B$, banks' liquidation decisions are not relevant for sufficiently poorly capitalized banks as the fragility stemming from depositors' run decisions leads to more liquidation than what a bank would prefer.

The case for better capitalized banks, which are only exposed to fundamental runs, is different. As shown in Section 5, fundamental runs induce the bank to partially liquidate its investment to meet depositors' withdrawals. Lemma 2 shows that banks with $k>\overline{k}_L$, and thus $\theta_L^B>\underline{\theta}$, liquidate their portfolios for $\theta \in \left[\underline{\theta}, \theta_L^B\right]$, even if no run has occurred. By contrast, those with $k < \overline{k}_L$ experience a run for any $\theta < \underline{\theta}$, with $\theta_L^B < \underline{\theta}$. It follows that the entire investment is liquidated for $\theta \in [0, \theta_L^B]$, while only partial liquidation takes place as a consequence of a run for $\theta \in [\theta_L^B, \underline{\theta}]$.

We can now analyze the extent to which evergreening occurs when runs are also possible. To this end, we compare the thresholds θ_L^B and $\theta^R = \{\underline{\theta}, \theta^*\}$ of banks' liquidation decisions and depositors' run behavior, respectively, with the liquidation threshold of the planner as given by θ_L^{SP} in (24). We have the following result.

Lemma 3 *In an economy with runs, $\theta_L^{SP} \geq \max\left\{\theta_L^B, \underline{\theta}\right\}$ for $1-k \leq L$ and $\theta^* > \theta_L^{SP} > \theta_L^B$ for $1-k > L$.*

The lemma shows that the early liquidation of the bank's project in the baseline economy is always inefficient. Highly capitalized banks with $1-k < L$ don't liquidate enough, thus carrying over until the final date projects that would be optimal to terminate at $t = 1$. The extent to which they engage in evergreening is captured by the difference $\theta_L^{SP} - \max\left\{\theta_L^B, \underline{\theta}\right\}$. When $\underline{\theta} > \theta_L^B$, fundamental runs force the liquidation of banks' projects when $\theta < \underline{\theta}$, while in the range $(\underline{\theta}, \theta_L^{SP})$ banks choose not to liquidate inefficient projects and evergreening occurs. When $\underline{\theta} < \theta_L^B$, evergreening occurs, instead, in the range $[\theta_L^B, \theta_L^{SP})$. By contrast, for low capital banks with $1-k > L$, there is always excessive liquidation resulting from panic runs, i.e., $\theta^* > \theta_L^{SP}$. Only when $1-k = L$ is a bank's liquidation decision efficient. The result is illustrated in Figure 5.

Insert Figure 5

We can now analyze the effect of the introduction of a first-loss guarantee x with full bankruptcy costs on the incidence of evergreening. We first characterize the bank liquidation threshold θ_{Lx}^B as given by the solution to

$$L - (1-k)r_2 = q\left(R(\theta + x) - (1-k)r_2\right),$$

and thus

$$\theta_{Lx}^B = \frac{L - (1-q)(1-k)r_2}{qR} - x = \theta_L^B - x.$$

Comparing θ_{Lx}^B with the run thresholds $\theta_x^R = \{\underline{\theta}_x, \theta_x^*\}$ in the presence of guarantees, as given in (14), it is easy to see that the same result as in Lemma 2 applies, i.e., $\theta_x^R > \theta_{Lx}^B$ for banks with $k < \overline{k}_L$.

We now compare early liquidation as described by $\max\left\{\theta_{Lx}^B, \theta_x^R\right\}$ to the planner's threshold θ_L^{SP}. In doing this, for the moment, we take bank underwriting effort q as fixed and not affected by the guarantee x.

Proposition 9 *The introduction of a first-loss loan guarantee with full bankruptcy costs has the following effect on evergreening incentives:*

a) When $1-k \leq L$*, the difference* $\theta_L^{SP} - \max\left\{\theta_{Lx}^B, \underline{\theta}_x\right\}$ *is larger than in the case without guarantees;*

b) When $1-k > L$*, there exists a level of capital* $\widetilde{k}_L \in [0, 1-L)$ *such that* $\theta_x^* \geq \theta_L^{SP}$ *for* $k \leq \widetilde{k}_L$ *and* $\theta_x^* < \theta_L^{SP}$ *for* $k > \widetilde{k}_L$*.*

The proposition, which is illustrated in Figure 5, shows that, holding q fixed, the presence of the loan guarantee worsens the evergreening problem for any level of bank capital. For highly capitalized banks, for which $1-k \leq L$, the guarantee increases the range of values of the fundamentals θ for which there is inefficient loan continuation. Banks exposed to panic runs, those with capital $\widetilde{k}_L < k < 1-L$, will now also evergreen loans as the guarantee reduces the panic run threshold to a value below the threshold for liquidation by the social planner, i.e., $\theta_L^{SP} - \theta_x^* > 0$ when $\widetilde{k}_L < k$.

The equilibrium effect of the loan guarantee on banks' evergreening incentives is, however, more complicated and may introduce a trade-off for the planner. This is due to the fact that q also changes with the introduction of the guarantee, thus affecting the occurrence of evergreening. Since we are interested in how loan guarantees affect the bank's overall incentives, to study this we consider the case where the decision to roll over a loan at date 1 is under the control of the bank rather than being determined by depositors' incentives to withdraw early. In other words, we focus on the region where $\theta_{Lx}^B \geq \underline{\theta}_x$. In this case, the measure of evergreening is given by $\theta_L^{SP} - \theta_{Lx}^B$ and is equal to

$$\theta_L^{SP} - \theta_{Lx}^B = \frac{1-q}{q}\frac{(1-k)r_2}{R} + x > 0.$$

While this difference is increasing in x, it is also decreasing in q. Thus, the negative effect associated with evergreening is at least partly offset by the positive underwriting incentive effect of the guarantee. These considerations raise the issue of how the two countervailing forces should be traded off by a social planner. To address this, we consider below how the introduction of a loan guarantee affects total output, which is defined as follows:

$$TO_x = \alpha \int_0^{\theta_{Lx}^B} L d\theta + \alpha \int_{\theta_{Lx}^B}^{1-x} qR\theta d\theta + \alpha \int_{1-x}^{1} qR\theta d\theta + (1-\alpha)\int_1^2 qR\theta d\theta - \frac{cq^2}{2} - 1. \tag{25}$$

We have the following result.

Proposition 10 *For small c, the introduction of a loan guarantee x increases total output:* $\frac{dTO_x}{dx} > 0$.

The proposition establishes that, as long as the bank's cost associated with its underwriting effort is not too large, total output increases with a loan guarantee, even though the bank makes more inefficient continuation decisions at date 1. The reason is that the positive underwriting incentive dominates the negative evergreening effect. From the perspective of the recent policy debate surrounding lenders' evergreening incentives, our results here suggest that even if at the margin the introduction of a loan guarantee may increase the extent of evergreening, overall economic output has the potential to increase as a result of the guarantee schemes put in place.

8 Extensions

In this section we extend the model in two directions. First, we extend the analysis to allow the date 2 interest rate on deposits, r_2, to change when a loan guarantee is put in place. Second, we study deposit insurance and compare it with the effects of loan guarantees. In what follows, for brevity we focus on the case of a first-loss loan guarantee when the transfer x is lost in bankruptcy, as in Section 6.1.

8.1 Deposit interest rates and loan guarantees

Throughout our analysis, we have assumed that the loan guarantee is introduced as a policy to stimulate lending in response to an unanticipated negative shock which makes lending riskier and less attractive for banks. As such, we have assumed that it is put in place after the bank has obtained funding. While we believe this represents the effect of policy responses to crisis episodes reasonably well, it is also likely true that downturns of longer duration, or loan guarantee programs that are longer-lived, may engender changes to deposit rates as banks and depositors recognize the presence of the guarantees when raising deposits. This case may be reflective of guarantee programs such as those used in mortgage markets, where government guarantees have long existed and are a normal part of the tools on which investors rely. Another case in point are loans provided to small businesses through the Small Business Administration (SBA) program, which guarantees loans under certain conditions for qualified lenders, and has as objective to stimulate lending.[15] It is useful, therefore, to discuss how allowing the deposit interest rate to adjust in the advent of the introduction of a loan guarantee may affect our results.

To study this issue, we modify the model slightly to allow the bank to change the date 2 deposit interest rate, r_2, after the loan guarantee is introduced. Specifically, we assume that when deposits

[15] See www.sba.gov/funding-programs/loans for details on the SBA program, and Brown and Earle (2017) as well as Bachas, Kim and Yannelis (2020) for studies on the stimulative effects of the SBA program.

are raised at date 0, the existence of any guarantee is common knowledge. The model is otherwise unchanged. We can now state the following result.

Proposition 11 *For all levels of bank capital k, the introduction of a first-loss loan guarantee x leads to a decrease in the date 2 deposit rate: $\frac{\partial r_2}{\partial x} < 0$.*

The proposition shows that banks respond to the introduction of the loan guarantee x by reducing the deposit interest rate r_2. Since this reduction in r_2 increases the profit accrued by the bank when its monitoring effort is successful, this translates into a higher effort q. In other words, given the complementarity in terms of the effects on bank incentives of the introduction of a loan guarantee and the pricing of deposit contracts, allowing the long term deposit rate to reflect the introduction of a loan guarantee further reinforces the improvement in underwriting incentives established in Section 6.1.

8.2 Deposit insurance

Our analysis has considered so far only guarantees that insure bank loans against default risk by borrowers. However, bank deposits are typically protected by other guarantees (i.e., deposit insurance). Such guarantees also contain a stimulative component since, in addition to reducing the required interest rate that must be paid to depositors (Cordella et al., 2018), they also directly increase stability by reducing depositors' run risk (see, e.g., Allen et al., 2018). In this section, we first show that deposit insurance differs substantially from loan guarantees in terms of the impact on bank monitoring incentives. Second, we confirm that the effect of loan guarantees remains unchanged in the presence of deposit insurance.

Following Allen et al. (2018), we consider a deposit guarantee scheme that ensures depositors always to receive a minimum repayment $\delta > 0$. To keep things simple, we assume $0 < \delta < \frac{L}{1-k}$ so that the deposit insurance is paid only to remaining depositors at date 2 whenever the bank does not have enough resources to pay them at least δ, thus making it comparable to the analysis with loan guarantees.[16]

8.2.1 An economy with only deposit insurance

As in the baseline model, we start by solving depositors' withdrawal decisions.

[16] The assumption that the guarantee is only paid at date 2 when depositors do not run is without loss of generality. As shown in Allen et al. (2018), the run threshold decreases in the guaranteed amount δ even when this is paid in the event of a run.

Proposition 12 *The run risk in the presence of deposit insurance depends on the level of bank capital, as follows:*

a) When $1-k \leq L$, fundamental runs occur for $\theta < \underline{\theta}_\delta(k) = \underline{\theta}(k)$, as given in (5).

b) When $1-k > L$, panic runs also occur for $\theta < \theta^*_\delta(q,k,\delta) < \theta^*(q,k)$ as given by

$$\theta^*_\delta(q,k,\delta) = \frac{(1-k)\,r_2}{R}\frac{(qr_2-\pi_1)+\delta\,(1-q)}{\left(qr_2-\pi_1\frac{1-k}{L}\right)+\delta\left(\frac{1-k}{L}-q\right)}, \tag{26}$$

where $\pi_1 = \int_0^{\overline{n}} dn + \int_{\overline{n}}^1 \frac{L}{(1-k)n}dn$. The threshold $\theta^*_\delta \in (\underline{\theta},1)$ decreases with q and δ: $\frac{\partial\theta^*_\delta(q,k)}{\partial q} < 0$ and $\frac{\partial\theta^*_\delta(q,k)}{\partial\delta} < 0$.

The run threshold is as in the baseline framework for banks with $1-k \leq L$, but is smaller for those with $1-k > L$. The reason is that the transfer δ increases what depositors expect to receive at date 2 only for the latter case, thus reducing their incentives to withdraw prematurely in the presence of panic runs. This contrasts with the result obtained in the presence of loan guarantees, where the run threshold is reduced also for well capitalized banks.

Given depositors' withdrawal decisions, we now analyze how deposit insurance affects bank underwriting standards. Similarly to above, denoting as $\theta^R_\delta \equiv \{\underline{\theta}_\delta, \theta^*_\delta\}$ the relevant run threshold, each bank chooses q to maximize

$$\alpha q\int_0^{\theta^R_\delta}\max\left\{0, R\theta\left(1-\frac{1-k}{L}\right)\right\}d\theta+\alpha q\int_{\theta^R_\delta}^1\left[R\theta-(1-k)\,r_2\right]d\theta+(1-\alpha)\,q\int_1^2\left[R-(1-k)\,r_2\right]d\theta-\frac{cq^2}{2}. \tag{27}$$

The interpretation of the terms in the expression for bank profits is as in the baseline framework. Importantly, and differently from the case of loan guarantees, the presence of deposit insurance does not directly increase the payoff that the bank obtains at date 2. However, banks benefit indirectly since it reduces their exposure to runs. We have the following result.

Proposition 13 *The introduction of a deposit guarantee scheme has no impact on bank monitoring effort when $1-k \leq L$, while it reduces it when $1-k > L$: $\frac{dq_\delta}{d\delta} = 0$ when $1-k \leq L$ and $\frac{dq^*_\delta}{d\delta} < 0$ when $1-k > L$.*

As the proposition shows, highly capitalized banks with $1-k \leq L$ are not affected by the introduction of the deposit insurance since the run threshold $\underline{\theta}_\delta$ does not depend on δ. By contrast, poorly capitalized banks with $1-k > L$ reduce their monitoring effort. For these banks, the introduction of deposit insurance reduces both depositors' incentives to run and the sensitivity of the run threshold to changes in the monitoring effort, with the latter effect dominating and leading to a reduced monitoring effort.

Overall, the result in Proposition 13 highlights the difference between deposit insurance and loan guarantees. In line with the idea that insurance mechanisms induce moral hazard considerations, the former never improves bank underwriting incentives. By contrast, the latter improve bank monitoring incentives, except for the most poorly capitalized banks when the transfer is protected from bankruptcy.

8.2.2 An economy with deposit insurance and loan guarantee

We now analyze the introduction of loan guarantees when bank deposits are insured. As before, we consider that depositors always obtain at least δ when the bank is unable to make the promised repayment.

We have the following result concerning depositors' withdrawal decisions.

Proposition 14 *The run risk in the presence of deposit insurance and first-loss loan guarantee depends on the level of bank capital as follows:*

a) When $1-k \leq L$, fundamental-driven runs occur for $\theta < \underline{\theta}_{\delta x}(k) = \underline{\theta}_x$, as characterized in Section 6.1.

b) When $1-k > L$, panic runs also occur for $\theta < \theta^*_{\delta x}(q,\delta) < \theta^*_x$, with

$$\theta^*_{\delta x}(q,\delta) = \theta^*_\delta - x, \tag{28}$$

where θ^*_δ is as in Proposition 12. The threshold $\theta^*_{\delta x} \in (\underline{\theta}_{\delta x}, 1-x)$ decreases with q, x, and δ: $\frac{\partial\theta^*_\delta(q,\delta)}{\partial q} < 0$, $\frac{\partial\theta^*_\delta(q,\delta)}{\partial x} < 0$ and $\frac{\partial\theta^*_\delta(q,\delta)}{\partial \delta} < 0$.

We can now study the impact of the loan guarantees on bank monitoring effort q.

Proposition 15 *In the presence of deposit insurance, the introduction of a first-loss loan guarantee with full bankruptcy costs always leads to an increase in bank monitoring effort: $\frac{d\underline{q}_{\delta x}}{dx} > 0$ and $\frac{dq^*_{\delta x}}{dx} > 0$.*

As shown in the proposition, the presence of deposit insurance does not alter the effect that the loan guarantees has on bank underwriting incentives, which remains beneficial for all banks irrespective of their level of capital.

9 Conclusions

In this paper, we present a model in which banks raise demandable deposits and grant long-term loans. A bank's expected return depends on the economy's fundamentals as well as on the bank's underwriting efforts. Our focus is on analyzing how the introduction of loan guarantees affects bank incentives and financial fragility. We show that, contrary to common wisdom, loan guarantees improve bank monitoring incentives except for the most poorly capitalized banks when the guaranteed amounts accrue to depositors even in the case when the bank's monitoring is unsuccessful. We also show, however, that the introduction of loan guarantees worsen banks' incentives to continue inefficient projects.

The issues studied here are germane to the policy debate concerning the use of public guarantee schemes to support bank lending during a crisis, or to enhance access to credit to particular sectors of the economy. Our results suggest that the perceived wisdom surrounding guarantee programs, such as those designed to protect retail depositors, may not translate to other types of guarantee schemes and, in particular, to loan guarantees. We therefore provide a novel lens through which loan guarantee schemes may be viewed, and policy initiatives evaluated.

We focus the analysis on the impact of loan guarantees when lenders' liability structures make them susceptible to runs, as is the case for banks. As discussed in Section 2, however, in some jurisdictions, such as the US, loan guarantees are also provided to non-bank lenders. We believe that our results concerning the effect of loan guarantees on a lender's effort should still be valid in this context as long as underwriting/monitoring is an important part of what these lenders do. Additionally, to the extent that nonbank institutions may be susceptible to some degree of rollover risk, the feedback effect between the lender's liabilities and their underwriting decisions should continue to hold as well.

In our setting, banks maximize their expected profits to remunerate their inside capital. This allows us to study the role of capital for banks, and how that impacts both banks' effort decisions and depositors' run choices, an issue that for the most part has been absent in the financial fragility literature (e.g., Diamond and Dybvig, 1983, and subsequent literature). In doing so, however, we take banks' capital structures as given. An interesting avenue for future research would be to endogenize bank capital structure and analyze how this interacts with bank lending standards and the threat of runs, as well as with loan guarantees. Carletti at al. (2022) move in this direction and study the feedback effects between banks' capital structure decisions and their choices concerning lending standards in a framework where depositors' withdrawal decisions are also endogenous.

10 References

Acharya, V V, L Borchert, M Jager, and S Steffen (2020), "Kicking the can down the road: government interventions in the European banking sector", *Journal of Financial Economics* (forthcoming).

Acharya, V V, M Crosignani, T Eisert, and C Eufinger (2020), “Zombie credit and (dis)inflation: Evidence from Europe”, NBER Working Paper No. 27158.

Acharya, V.V. and N. Mora (2015), “A crisis of banks as liquidity providers”, *Journal of Finance*, 70(1), 1-43.

Ahnert, T. and M. Kuncl (2021), "Government Loan Guarantees, Market Liquidity and Lending Standards", working paper.

Allen, F., E. Carletti, and R. Marquez (2011), “Credit market competition and capital regulation,” *Review of Financial Studies*, 24(4), 983-1018.

Allen, F., E. Carletti, I. Goldstein and A. Leonello (2018), “Government Guarantees and Financial Stability” *Journal of Economic Theory*, 177, 518-557.

Altavilla, C., Ellul, A., Pagano, M., Polo, A., and T. Vlassopoulos (2021), "Loan guarantees, bank lending and credit risk reallocation ", Center for Financial Studies Working Paper 672.

Artavanis, N., D. Paravisini, C. Robles-Garcia, A. Seru, and M. Tsoutsoura (2019), "Deposit withdrawals", unpublished working paper.

Bachas, N., Kim, O.S., and C. Yannelis (2020) "Loan guarantees and credit supply", *Journal of Financial Economics*, 179, 872-894.

Balyuk, T., N.R. Prabhala, and M. Puri (2020), “Indirect costs of government aid and intermediary supply effects: Lessons from the paycheck protection program, NBER Working paper 29114.

Bartik, A.W., Cullen, Z. B., Glaeser, E. L., Luca, M., Stanton, C. T. and Sunderam, A. (2020), "The targeting and impact of Paycheck Protection Program loans to small businesses. NBER working paper 27623.

Baudino, P. (2020), “Public Guarantees for Bank Lending in Response to the Covid-19 Pandemic,” FSI Briefs, April. https://www.bis.org/fsi/fsibriefs5.htm.

Beyhaghi, M. (2021), "Third-Party Credit Guarantees and The Cost of Debt: Evidence from Corporate Loans", working paper.

Blattner, L., L. Farinha, and F. Rebelo (2021), "When Losses Turn Into Loans: The Cost of Weak Banks", *American Economic Review* (forthcoming).

Brown, J.D, and J.S. Earle (2017), "Finance and growth at the firm level: Evidence from SBA loans", *Journal of Finance*, 72(3), 1039-1080.

Bruche, M., and G. Llobet (2014), "Preventing zombie lending", *Review of Financial Studies,* 27(3), 923-956.

Caballero, R.J., T. Hoshi, and A.K. Kashyap (2008), “Zombie lending and depressed restructuring in Japan”, *American Economic Review,* 98(5), 1943–77.

Calomiris, C. (1990), “Is deposit insurance necessary? A historical perspective”, *Journal of Economic History*, 50(2), 283-295.

Calomiris, C.W., and C.M., Kahn (1991), “The role of demandable debt in structuring optimal banking arrangements”, *American Economic Review*, 81(3),497-513.

Carletti, E., F. De Marco, V. Ioannidou and E. Sette (2020), "Banks as Patient Lenders: Evidence from a Tax Reform", forthcoming, *Journal of Financial Economics.*

Carletti, E., A. Leonello and R. Marquez (2022), "Liquidity provision without liquidity insurance: The role of bank capital", mimeo.

Chodorow-Reich, G., Darmouni, O., Luck, S., and Plosser, M. (2021), "Bank Liquidity Provision across the Firm Size Distribution," Forthcoming in *Journal of Financial Economics.*

Cole, A. (2020), "The Impact of the Paycheck Protection Program on Small Businesses: Evidence from Administrative Payroll Data", working paper.

Cordella, T., G. Dell'Ariccia, and R. Marquez (2018), "Government guarantees, transparency, and bank risk taking," *IMF Economic Review*, 66, 116-143.

Core, F. and F. De Marco (2020), "Public Guarantees for Small Businesses in Italy during Covid-19", working paper, Bocconi University.

Dell'Ariccia G. and R. Marquez (2006), "Competition among regulators and credit market integration," *Journal of Financial Economics*, 79, 401-30.

Dell'Ariccia G., L. Laeven, and R. Marquez (2014), "Real interest rates, leverage, and bank risk-taking," *Journal of Economic Theory*, 149, 65-99.

Diamond, D.W. (1991), "Debt maturity structure and liquidity risk", *Quarterly Journal of Economics*, 106(3), 709-737.

Diamond, D.W. and P.H. Dybvig (1983), "Bank runs, deposit insurance and liquidity", *Journal of Political Economy*, 91(3), 401-419.

Diamond, D. W., and R. Rajan (2000), "A theory of bank capital ", *Journal of Finance*, 55(6), 2431-2465.

Diamond, D.W. and R. Rajan (2001), "Liquidity Risk, Liquidity Creation, and Financial Fragility: A Theory of Banking", *Journal of Political Economy,* 109(2), 287-327.

Ding, W., Levine, R., Lin, C., and Xie, W. (2020), "Corporate Immunity to the COVID-19 Pandemic," Forthcoming in *Journal of Financial Economics.*

Duchin, R., X. Martin and R. Michaely (2020), "Concierge treatment from banks: Evidence from the Paycheck Protection Program," working paper.

Dursun-de Neef, H.O. and A. Schandlbauer (2021), " COVID-19 and lending responses of European banks", working paper, Goethe University Frankfurt.

ECB (2020), "Public loan guarantees and bank lending in the COVID-19 period", ECB Economic Bulletin 6/2020.

Egan, M., Hortaçsu, A., and G. Matvos (2017), "Deposit competition and financial fragility: Evidence from the us banking sector". *American Economic Review*, 107(1), 169-216.

Eichenbaum, M., Rebelo, S., and M. Trabandt (2020), "The Macroeconomics of Epidemics," Forthcoming in *Review of Financial Studies.*

Eisenbach, T. (2017), "Rollover risk as market discipline: A two-sided inefficiency", *Journal of Financial Economics,* 126(2), 252-269.

Erel, I., and Liebersohn, J. (2022), "Can FinTech reduce disparities in access to finance? Evidence from the Paycheck Protection Program", *Journal of Financial Economics*, 146(1), 90-118.

European Central Bank (2022), "Financial integration and structure in the euro area", April 2022, available at https://www.ecb.europa.eu/pub/pdf/fie/ecb.fie202204~4c4f5f572f.en.pdf.

European Commission (2020), "Temporary framework for State aid measures to support the economy in the current COVID-19 outbreak", communication 2020/C 91 I/01.

European Commission (2022), "State aid: Commission adopts Temporary Crisis Framework to support the economy in the context of Russia's invasion of Ukraine",

https://ec.europa.eu/commission/presscorner/detail/en/STATEMENT_22_1949.

Falagiarda, M., Prapiestis, A., and E. Rancoita (2020), “Public Loan Guarantees and Bank Lending in the COVID-19 Period,” in: *ECB Economic Bulletin*,6/2020.

Flannery, M. J. (1986), "Asymmetric information and risky debt maturity choice", *Journal of Finance*, 41(1), 19-37.

Goldstein I. (2012), “Empirical literature on financial crises: fundamentals vs. panic”, in: Caprio, G. (Ed.), *The Evidence and Impact of Financial Globalization*, Elsevier, Amsterdam, pp. 523–534.

Goldstein, I., and Pauzner, A., (2005), "Demand deposit contract and probability of bank runs", *Journal of Finance*, 60(3), 1293-1328.

Gonzalez-Uribe, J. and S. Wang (2020), “The Effects of Small-Firm Loan Guarantees in the UK: Insights for the COVID-19 Pandemic Crisis,” working paper.

Gopal, M. and P. Schnabl (2022), "The rise of finance companies and FinTech lenders in small business lending", *Review of Financial Studies,* forthcoming.

Granja, J., Makridis, C., Yannelis, C. and E. Zwick (2022), "Did the Paycheck Protection Program Hit the Target?", *Journal of Financial Economics*, 145(3), 725-761.

Griffin, J. M., Kruger, S., & Mahajan, P. (2022), "Did FinTech lenders facilitate PPP fraud?", *Journal of Finance*, forthcoming.

Gropp, R., Gruendl, C., and Guettler, A. (2014), “The Impact of Public Guarantees on Bank Risk-Taking: Evidence from a Natural Experiment”, *Review of Finance*, 18, 457–488

Hellmann, T., K. Murdock, and J. Stiglitz (2000), “Liberalization, moral hazard in banking, and prudential regulation: Are capital requirements enough?”, *American Economic Review*, 90, 147-65.

Holmstrom, B. and J. Tirole (1997), “Financial intermediation, loanable funds, and the real sector,” *Quarterly Journal of Economics*, 112, 663-691.

Howell, S. T., Kuchler, T., and Stroebel, J. (2020), "Which lenders had the highest minority share among their Payment Protection Program (PPP) loans?", working paper.

Hu Y., and F. Varas (2021), "A theory of zombie lending", *Journal of Finance* (forthcoming).

Hubbard, G. and Strain, M. R. (2020), "Has the Paycheck Protection Program Succeeded?", *Brookings Papers of Economic Activity* (forthcoming).

Iyer, R., and M. Puri (2012), "Understanding bank runs: the importance of depositor-bank relationships and networks", *American Economic Review*, 102, 1414–1445.

Iyer, R., M. Puri and N. Ryan (2016), "A tale of two runs: depositor responses to bank solvency risk", *Journal of Finance,* 71, 2687-2725.

James, C. (1991), "The Losses Realized in Bank Failures", *Journal of Finance,* 46(4), 1223-1242.

Jiménez, G., Laeven, L., Martinez-Miera, D., and J.L. Peydró (2022), "Public Guarantees, Relationship Lending and Bank Credit: Evidence from the COVID-19 Crisis", unpublished manuscript.

Kashyap, A., Tsomocos, D., and A. Vardoulakis (2019), "Optimal bank regulation in the presence of credit and run risk", working paper 2017-097, Board of Governors of the Federal Reserve System.

Keeley, M.C. (1990), "Deposit insurance, risk and market power in banking", American Economic Review, 80(5), 1183-1200

Keister, T. (2016), "Bailouts and financial fragility", *Review of Economic Studies,* 83(2), 704-736.

Kelly, B., Lustig, H., and Van Nieuwerburgh, S. (2016), "Too-Systemic-to-Fail: What Option Markets Imply about Sector-Wide Government Guarantees," *American Economic Review*, 106, 1278–1319.

Laeven, L., Schepens, G. and I. Schnabel (2020), "Zombification in Europe in times of pandemic", VoxEu.org October 2020.

Li, L., Strahan, P. E., and Zhang, S., 2020, "Banks as Lenders of First Resort: Evidence from the COVID-19 Crisis," *Review of Corporate Finance Studies*, 9, 472–500.

Marcus, A. (1984), "Deregulation and bank financial policy", *Journal of Banking and Finance*, 8, 557-565.

Martin, C., Puri, M. and A. Ufier (2018) "Deposit inflows and outflows in failing banks: the role of deposit insurance", NBER working paper 24589.

Mehran, H., and A. Thakor (2011), "Bank capital and value in the cross-section," *Review of Financial Studies*, 24(4), 1019-67.

Morrison, A., and L. White (2005), "Crises and capital requirements in banking," *American Economic Review*, 95, 1548-72.

Parlour, C. and A. Winton (2013), "Laying off credit risk: Loan sales versus credit default swaps", *Journal of Financial Economics*, 107, 25-45.

Peek, J., and E.S. Rosengren (2005), "Unnatural Selection: Perverse incentives and the misallocation of credit in Japan", *American Economic Review,* 95(4), 1144–1166.

Repullo, R. (2004), "Capital requirements, market power, and risk-taking in banking," *Journal of Financial Intermediation*, 13, 156-82.

Schivardi, F., E. Sette, and G. Tabellini (2021), "Credit misallocation during the European financial crisis", *Economic Journal (*forthcoming).

Thakor, A. (2014), "Bank Capital and Financial Stability: An Economic Trade-Off or a Faustian Bargain?", *Annual Review of Financial Economics*, 6, 185-223.

11 Appendix A

Proof of Proposition 1: The proof makes use of the technical approach developed in Goldstein and Pauzner (2005) since, like theirs, our model also exhibits the property of one-sided strategic complementarity, i.e., a depositor 's incentive to run does not monotonically increases with the proportion of depositors running.

We proceed in steps. First, we pin down the threshold $\underline{\theta}(k)$, which corresponds to the upper bound of the lower dominance region, as characterized in the main text. Second, we characterize the threshold $\theta^*(q, k, L, r_2)$, which summarizes depositors' withdrawal decision in the intermediate range of fundamentals, i.e., when $\theta \in [\underline{\theta}(k), \overline{\theta})$. Third, we show that for any $1-k \leq L$, the relevant run threshold is $\underline{\theta}(k)$, while it is $\theta^*(q, k, L, r_2) > \underline{\theta}(k)$ for any $1-k > L$. We conclude the proof with the comparative statics for the two run thresholds with respect to q, L, and k.

Solving (2) with respect to θ, we obtain the threshold $\underline{\theta}(k)$ as given in (5) in the proposition. Given the definition of the lower dominance region, when $\theta < \underline{\theta}(k)$, depositors always find it optimal to withdraw irrespective of what others do. Symmetrically, given the definition of the upper dominance region, depositors find it optimal to wait until date 2 when $\theta > \overline{\theta}$. Given that $\widehat{\theta} \leq 1-2\varepsilon$ and $\overline{\theta} = \widehat{\theta}$, as shown in the characterization of the upper dominance region, the relevant range of θ from the perspective of depositors choosing whether to withdraw is $[0, 1-2\varepsilon)$. It follows that the discontinuity in the distribution of θ does not play a role for depositors' decisions.

For $\theta \in [\underline{\theta}(k), \overline{\theta})$, a depositor's withdrawal decision depends on what other depositors do when $1-k > L$. The arguments in the proof of Theorem 1 in Goldstein and Pauzner (2005) establish that there is a unique equilibrium in which depositors run if and only if the signal they receive is below a common signal s^*. A depositor who receives the signal s^* is exactly indifferent between withdrawing at dates 1 and 2.

To characterize the threshold signal s^*, we start by assuming that all depositors behave according to the threshold strategy s'. Then, the fraction of depositors withdrawing at date 1, $n(\theta, s')$, is equal to the probability of receiving a signal below s' and can be specified as follows:

$$n\left(\theta, s'\right) = \begin{cases} 1 & \text{if } \theta \leq s' - \varepsilon \\ \frac{s'-\theta+\varepsilon}{2\varepsilon} & \text{if } s'-\varepsilon < \theta \leq s'+\varepsilon \\ 0 & \text{if } \theta > s'+\varepsilon \end{cases} .$$

Depositors' withdrawal decisions are characterized by the pair $\{s^*, \theta^*\}$, which corresponds to the solution to the following system of equations:

$$R\theta^* \left(1 - \frac{n\left(\theta^*, s^*\right)(1-k)}{L}\right) - \left(1 - n\left(\theta^*, s^*\right)\right)(1-k)\, r_2 = 0, \tag{29}$$

and

$$\Delta\left(s^*, n\left(\theta, s^*\right)\right) = q r_2 \Pr\left(\theta > \theta^* \middle| s^*\right) - 1 \Pr\left(\theta > \theta_n \middle| s^*\right) - \frac{L}{(1-k)\, n\left(\theta, s^*\right)} \Pr\left(\theta < \theta_n \middle| s^*\right) = 0, \tag{30}$$

where $\theta_n = s^* + \varepsilon - 2\varepsilon \frac{L}{1-k}$ represents the level of θ for which the bank liquidates the entire portfolio at date 1 and, thus, is equal to the solution to

$$n\left(\theta_n, s^*\right)(1-k) = L.$$

Condition (29) identifies the level of fundamental, θ^*, at which the bank is at the brink of insolvency at date 2 when $n(\theta^*, s^*) > 0$ depositors run, for given s^*. Condition (30) is an indifference condition for a depositor that receives a signal exactly equal to the threshold signal s^*: the first term represents his expected utility from withdrawing at date 2, while the second and third terms represent the expected utility from withdrawing at date 1. This condition pins down s^* given $\theta^*(s^*)$ from (29), so that together the two equations characterize the equilibrium withdrawal decisions $\{s^*, \theta^*\}$. In other words, the equilibrium threshold signal s^* corresponds to the signal at which the expression (30) is equal to zero, i.e., $\Delta(s^*, n(\theta, s^*)) = 0$.

The function $\Delta(s_i, n(\theta, s'))$ representing a depositor' utility differential for any signal s_i when all other depositors behave accordingly to the threshold strategy s' exhibits the same properties as the corresponding function in Goldstein and Pauzner (2005); thus, the arguments in their proof can be applied to show that our model has a unique threshold equilibrium. First, $\Delta(s_i, n(.))$ is continuous in s_i, negative when $s' < \underline{\theta}(k) - \varepsilon$ and positive when $s' > \overline{\theta} + \varepsilon$ because of the definition of the lower and upper dominance regions. To see this, we can rearrange the LHS in (29) as follows:

$$R\theta - (1-k) r_2 - n(\theta, s^*) \left(R\theta \frac{(1-k)}{L} - (1-k) r_2 \right),$$

so that it is easy to see that the expression in (29) is always negative when θ falls in the lower dominance region and positive when it is in the upper dominance region. Since $\Pr(\theta > \theta^* | s')$ is then 0 and 1 in these two extreme regions of fundamental, this also implies that a depositor's expected utility differential between withdrawing at dates 2 and 1 is also negative when $s' < \underline{\theta}(k) - \varepsilon$ and positive when $s' \geq \overline{\theta} + \varepsilon$.

Second, $\Delta(s_i, n(.))$ is non-decreasing when both the individual signal s_i and the threshold strategy s' shift upward. Formally, take $h > 0$, $\Delta(s_i + h, n(\theta + h, s'))$ is non-decreasing in h and strictly increasing when there is a positive probability that $n < \overline{n}$ in the range $[s' - \varepsilon, s' + \varepsilon]$ and $s' < \overline{\theta} + \varepsilon$. This is because an increase in h leads to a shift of equal magnitude in s' and s_i, which leaves $n(.)$ unaffected, while it is associated with a better θ. To see this, differentiating (29) with respect to θ keeping n constant, we obtain

$$R\left(1 - \frac{n(\theta, s^*)(1-k)}{L}\right) > 0.$$

Hence, it follows that, in the presence of an equal positive shift in the individual signal and threshold signal, $\Pr(\theta > \theta^* | s')$ strictly increases and so does the expected utility differential.

All these properties imply that there is a unique s^* satisfying $\Delta(s^*, n(\theta, s^*)) = 0$ and also that $\Delta(s_i, n(\theta, s^*)) < 0$ if $s_i < s^*$ and $\Delta(s_i, n(\theta, s^*)) > 0$ if $s_i > s^*$. To obtain the expression for $\theta^*(q, k, L, r_2)$ as in the proposition, we perform a change of variable by defining $\theta^*(n) = s^* + \varepsilon(1 - 2n)$. At the limit when $\varepsilon \to 0$, $\theta^*(n) \to s^*$ and we denote the run threshold as $\theta^*(q, k, L, r_2)$, which corresponds to the solution to

$$\int_0^{\widehat{n}(\theta)} q r_2 dn - \int_0^{\overline{n}} dn - \int_{\overline{n}}^1 \frac{L}{(1-k) n} dn = 0, \tag{31}$$

where $\widehat{n}(\theta^*)$ solves (4) and $\overline{n}$ solves

$$(1-k)n = L.$$

The expression in (6) is obtained by rearranging the terms, using $\underline{\theta} = \frac{(1-k)r_2}{R}$, and denoting

$$\pi_1 = \int_0^{\overline{n}} dn + \int_{\overline{n}}^1 \frac{L}{(1-k)\,n} dn. \tag{32}$$

Now, we move on to show that the relevant run threshold is $\underline{\theta}(k)$ when $1-k \leq L$ and $\theta^*(q,k,L,r_2)$ when $1-k > L$. Consider first the case in which $1-k \leq L$. When $1-k = L$, $\pi_1 = 1$ and (29) simplifies to

$$(1 - n(\theta^*, s^*))\left[R\theta - (1-k)\,r_2\right],$$

which is positive for $\theta > \underline{\theta}(k)$ and negative for $\theta < \underline{\theta}(k)$ for any $n(\theta^*, s^*) < 1$. Then, from (30), it follows that running is optimal when $\theta < \underline{\theta}(k)$, irrespective of $n(.)$. Hence, the relevant run threshold is $\underline{\theta}(k)$ when $1-k = L$. Since $\underline{\theta}(k)$ is decreasing in $1-k$, condition (29) becomes less binding for any n when $1-k$ falls below L. This implies that $\underline{\theta}(k)$ is still the relevant run threshold when $1-k < L$.

Consider now the case where $1-k > L$. Differentiating (29) with respect to θ, we obtain

$$R\left(1 - \frac{n(\theta, s^*)(1-k)}{L}\right) - \frac{\partial n(\theta, s^*)}{\partial \theta}\left[R\theta\frac{(1-k)}{L} - (1-k)\,r_2\right] > 0,$$

for any $\theta > \underline{\theta}(k)$ when $1-k > L$ and $\frac{\partial n(\theta,s^*)}{\partial\theta} < 0$. It follows that $\theta^*(q,k,L,r_2) > \underline{\theta}(k)$ when $1-k > L$.

To complete the proof, we compute $\frac{\partial\underline{\theta}(k)}{\partial k}$, as well as $\frac{\partial\theta^*(q,k,L,r_2)}{\partial q}$, $\frac{\partial\theta^*(q,k,L,r_2)}{\partial L}$, and $\frac{\partial\theta^*(q,k,L,r_2)}{\partial k}$. Differentiating (5) with respect to k, we obtain $\frac{\partial\underline{\theta}(k)}{\partial k} = -\frac{r_2}{R} < 0$. Using (6), we compute the effect of q, L, and k on $\theta^*(q,k,L,r_2)$ as follows:

$$\begin{aligned}\frac{\partial\theta^*(q,k,L,r_2)}{\partial q} &= \frac{\underline{\theta}}{\left(qr_2 - \pi_1\frac{(1-k)}{L}\right)^2}\left\{r_2\left(qr_2 - \pi_1\frac{(1-k)}{L}\right) - r_2\,(qr_2 - \pi_1)\right\} \\ &= -\frac{\underline{\theta}r_2\pi_1}{\left(qr_2 - \pi_1\frac{(1-k)}{L}\right)^2}\left[\frac{(1-k)}{L} - 1\right] < 0,\end{aligned}$$

$$\frac{\partial\theta^*(q,k,L,r_2)}{\partial L} = \frac{\underline{\theta}}{\left(qr_2 - \pi_1\frac{(1-k)}{L}\right)^2}\left\{-\frac{\partial\pi_1}{\partial L}\left(qr_2 - \pi_1\frac{(1-k)}{L}\right) + (qr_2 - \pi_1)\frac{(1-k)}{L}\left[\frac{\partial\pi_1}{\partial L} - \frac{\pi_1}{L}\right]\right\} < 0,$$

and

$$\frac{\partial\theta^*(q,k,L,r_2)}{\partial k} = \frac{1}{\left(qr_2 - \pi_1\frac{(1-k)}{L}\right)}\left\{\frac{\partial\underline{\theta}}{\partial k}(qr_2 - \pi_1) - \underline{\theta}\frac{\partial\pi_1}{\partial k} + \frac{\theta^*}{L}\left[\frac{\partial\pi_1}{\partial k}(1-k) - \pi_1\right]\right\} < 0,$$

with $\frac{\partial \pi_1}{\partial L} = \int_{\overline{n}}^{1} \frac{1}{(1-k)n} dn > 0$, $\frac{\partial \pi_1}{\partial k} = \int_{\overline{n}}^{1} \frac{L}{(1-k)^2 n} dn > 0$, and $\frac{\partial \pi_1}{\partial L} - \frac{\pi_1}{L} = -\frac{1}{L}\int_0^{\overline{n}} dn$. Hence, the proposition follows. $\square$

Proof of Proposition 2: Using backward induction, we first compute the optimal q and then solve for r_2. Concerning the choice of q, (10) and (11) are obtained by differentiating (7) with respect to q, setting $\theta^R = \underline{\theta}$ when $1-k \leq L$ and $\theta^R = \theta^*$ when $1-k > L$, respectively.

We now move to the choice of r_2. Consider first the case when $1-k \leq L$ when the relevant run threshold is $\underline{\theta}$. Since $\frac{\partial \underline{\theta}}{\partial r_2} > 0$ and a higher r_2 reduces bank's profits when no runs occur, it is optimal for the bank to choose the lowest possible r_2, which corresponds to the solution of (8) holding with equality.

Consider now the case when $1-k > L$. In this case, the above argument does not apply since $\frac{\partial \theta^*}{\partial r_2} < 0$ may hold. The derivative $\frac{\partial \theta^*}{\partial r_2}$ is obtaining differentiating (6) with respect to r_2 and it is given by

$$\frac{\partial \theta^*}{\partial r_2} = \frac{1}{R}\frac{qr_2 - \pi_1}{qr_2 - \pi_1 \frac{(1-k)}{L}} - \frac{r_2}{R}\frac{q\pi_1\left[\frac{1-k}{L} - 1\right]}{\left(qr_2 - \pi_1 \frac{(1-k)}{L}\right)^2},$$

whose sign is potentially ambiguous. It is easy to see from (7) that Π is strictly decreasing in r_2 when $\frac{\partial \theta^*}{\partial r_2} > 0$. Hence, assuming it is consistent with (8) to hold, banks will always be better off choosing r_2 in the range where $\frac{\partial \theta^*}{\partial r_2} < 0$ that is, in other words, $\frac{\partial \theta^*}{\partial r_2} < 0$ in equilibrium.

We write the Lagrangian for the bank's problem as

$$\mathcal{L} = \Pi|_{q=q^*} - \mu\left\{1 - \alpha\int_0^{\theta^*}\frac{L}{1-k}d\theta - \alpha\int_{\theta^*}^1 qr_2 d\theta - (1-\alpha)\int_1^2 qr_2 d\theta\right\},$$

where Π is given in (7). The Kuhn-Tucker conditions are

$$\begin{aligned} &-\alpha\frac{\partial \theta^*}{\partial r_2}\left[R\theta^* - (1-k)r_2\right] - \alpha\int_{\theta^*}^1 q^*(1-k)\,d\theta - (1-\alpha)\int_1^2 q^*(1-k)\,d\theta + \frac{\partial \Pi}{\partial q}\frac{dq^*}{dr_2} + \alpha\mu\int_{\theta^*}^1 q^* d\theta \\ &+ (1-\alpha)\mu\int_1^2 q^* d\theta - \alpha\mu\left[\frac{\partial \theta^*}{\partial r_2} + \frac{\partial \theta^*}{\partial q}\frac{dq^*}{dr_2}\right]\left[q^* r_2 - \frac{L}{1-k}\right] = 0, \end{aligned} \tag{33}$$

$$\mu\left\{1 - \alpha\int_0^{\theta^*}\frac{L}{1-k}d\theta - \alpha\int_{\theta^*}^1 qr_2 d\theta - (1-\alpha)\int_1^2 qr_2 d\theta\right\} = 0,$$

$$\mu \geq 0.$$

The derivative $\frac{dq^*}{dr_2}$ is obtained using the implicit function theorem.

When $\mu = 0$, $1 - \alpha\int_0^{\theta^*}\frac{L}{1-k}d\theta - \alpha\int_{\theta^*}^1 qr_2 d\theta - (1-\alpha)\int_1^2 qr_2 d\theta > 0$, i.e., (8) is not binding and r_2 solves (12) in the proposition. Since $1-k > L$ and $q^* \leq 1$, r_2 must be greater than 1 for (8) to be satisfied. When $\mu > 0$, bank profit decreases with r_2 and therefore the bank will choose the lowest level of r_2 that solves

$$1 - \alpha \int_0^{\theta^*} \frac{L}{1-k} d\theta - \alpha \int_{\theta^*}^{1} q r_2 d\theta - (1-\alpha) \int_1^2 q r_2 d\theta = 0.$$

The solution is again greater than 1 in order for (8) to hold. The Lagrange multiplier μ is then pinned down by (33) and is equal to

$$\mu = \frac{\alpha \frac{\partial \theta^*}{\partial r_2} \left[R\theta^* - (1-k) r_2\right] + \alpha \int_{\theta^*}^1 (1-k) \, d\theta + (1-\alpha) \int_1^2 (1-k) \, d\theta}{\alpha \int_{\theta^*}^1 q d\theta + (1-\alpha) \int_1^2 q d\theta - \alpha \left[\frac{\partial \theta^*}{\partial r_2} + \frac{\partial \theta^*}{\partial q} \frac{dq}{dr_2}\right] \left[q r_2 - \frac{L}{1-k}\right]}.$$

Hence, the proposition follows. □

Proof of Proposition 3: To characterize the run thresholds $\underline{\theta}_x$ and θ^*_x, we follow the same steps as in the proof of Proposition 1. We start characterizing the range of fundamentals in which running is a dominant strategy. The threshold $\underline{\theta}_x$ is the solution to

$$R(\theta + x) - (1-k) r_2 = 0,$$

and is equal to

$$\underline{\theta}_x = \frac{(1-k) r_2}{R} - x = \underline{\theta} - x.$$

For any $\theta < \underline{\theta}_x$, a depositor expects to receive 0 at date 2 and 1 at date 1 even if no depositors run, thus, it is always optimal to run in this range. The characterization of the upper dominance region is as in Section 4: Depositors finds it optimal to wait until date 2 when $\theta > \overline{\theta}$, with $\overline{\theta} \to 1$.

The key difference relative to the proof of Proposition 1 is that now for $\theta \in [1-x, \overline{\theta})$, the return accrued to the bank at date 2 is R rather than $R\theta$, which matters for the properties of the expected utility differential in the characterization of the run threshold θ^*_x in the case $1-k > L$.

Depositors' withdrawal decisions are fully characterized by the pair $\{s^*_x, \theta^*_x\}$ as given by the solution to the following system of equations.

$$R\theta^*_x \left(1 - \frac{n(\theta^*_x, s^*_x)(1-k)}{L}\right) - (1 - n(\theta^*_x, s^*_x))(1-k) r_2 = 0, \tag{34}$$

and

$$\Delta_x(s^*_x, n(.)) = q r_2 \Pr(\theta > \theta^*_x | s^*_x) - 1 \Pr(\theta > \theta_n | s^*_x) - \frac{L}{(1-k) n(\theta, s^*_x)} \Pr(\theta < \theta_n | s^*_x) = 0, \tag{35}$$

The meaning of the two equations is the same as in the proof of Proposition 1, with (34) pinning down the bank failure threshold θ^*_x and (35) identifying the threshold signal s^*_x at which a depositor's expected utility differential between withdrawing at date 2 and date 1, $\Delta_x(s_i, n(.))$, is exactly zero. The function $\Delta_x(s_i, n(.))$ satisfies the same properties as the corresponding function in the proof of Proposition 1. The only difference is that when a depositor receives a signal such that he expects θ to be in the range $[1-x, \overline{\theta})$, his expected utility differential is non-decreasing rather than strictly increasing in the signal s_i for any n. This results from the fact that in that range, due to the guarantee, the bank accrues R from the loan. Yet, considering a generic threshold signal s', the function is negative when $s' < \underline{\theta}(k) - \varepsilon$, positive when $s' \geq \overline{\theta} + \varepsilon$ and strictly increasing in the threshold signal s' when there is a positive probability that $n < \overline{n}$ and $s' < 1 - x - \varepsilon$. Since $\Delta_x(s_i, n(.))$ is constant in s' when $1 - x - \varepsilon \leq s' < \overline{\theta} + \varepsilon$, strictly positive when $s' \geq \overline{\theta} + \varepsilon$ and continuous, it follows that it crosses zero for $s' < 1 - x - \varepsilon$, i.e., $s_x^* < 1 - x - \varepsilon$ so that s_x^* is unique.

Following the same steps as in the proof of Proposition 1, considering the limit case $\varepsilon \to 0$, we can specify depositor's indifference condition as

$$\int_0^{\widehat{n}_x(\theta)} q r_2 d\theta = \pi_1, \tag{36}$$

where π_1 is given in (32) and $\widehat{n}_x(\theta) > \widehat{n}(\theta)$ corresponds to the solution to

$$R(\theta + x)\left(1 - \widehat{n}_x(\theta)\frac{(1-k)}{L}\right) - (1 - \widehat{n}_x(\theta))(1-k) r_2 = 0.$$

After a few manipulations, we obtain $\theta_x^* = \theta^* - x$. It follows immediately that $\theta_x^* < \theta^*$ for any $x > 0$, and $\frac{\partial \theta^*}{\partial x} = -1 < 0$. Condition (14) in the proposition is thus obtained simply combining together the case when $1 - k \leq L$ and when $1 - k > L$.

Using the same argument as in the proof of Proposition 1, we have that $\theta_x^* > \underline{\theta}_x$ and $\underline{\theta}_x$ is the run threshold when $1 - k \leq L$. It is easy to see that $\frac{\partial \underline{\theta}_x}{\partial x} = -1 < 0$ and $\frac{\partial \underline{\theta}_x}{\partial k} = -\frac{r_2}{R} < 0$. This completes the proof. □

Proof of Proposition 4: To compute the effect of x on $q_x^R \equiv \left\{\underline{q}_x, q_x^*\right\}$, we consider separately the case when $1 - k \leq L$ and when $1 - k > L$. We start from the former. Differentiating (16) with respect to x we obtain

$$\begin{aligned}
&-\alpha \frac{\partial \underline{\theta}_x}{\partial x}\left[R(\underline{\theta}_x + x) - (1-k) r_2 - R(\underline{\theta}_x + x)\left(1 - \frac{(1-k)}{L}\right)\right] \\
&+\alpha \int_0^{\underline{\theta}_x} R\left(1 - \frac{(1-k)}{L}\right) d\theta + \alpha \int_{\underline{\theta}_x}^{1-x} R d\theta \\
&= \alpha\left[R(\underline{\theta}_x + x)\frac{(1-k)}{L} - (1-k) r_2\right] + \alpha \int_0^{\underline{\theta}_x} R\left(1 - \frac{(1-k)}{L}\right) d\theta + \alpha \int_{\underline{\theta}_x}^{1-x} R d\theta,
\end{aligned} \tag{37}$$

since $\frac{\partial \underline{\theta}_x}{\partial x} = -1$. For banks with k such that $1-k = L$, $\frac{d\underline{q}_x}{dx} > 0$ since the expression above simplifies to $\alpha \int_{\underline{\theta}_x}^{1-x} R d\theta > 0$. The same applies to banks with $k = 1$ since (37) simplifies to

$$+\alpha \int_0^{\underline{\theta}_x|_{k=1}} R d\theta + \alpha \int_{\underline{\theta}_x|_{k=1}}^{1-x} R d\theta > 0.$$

For values of $k \in (1-L, 1)$, the expression in (37) can be rearranged as

$$+\alpha \left\{ R \left[x \frac{(1-k)}{L} + 1 - x \right] - (1-k) r_2 \right\}. \tag{38}$$

The expression above is linear in k. Hence, since (37) is linear, positive at $k = 1$ and $k = 1 - L$, it follows that it must also be positive for any $k \in (1-L, 1)$.

Consider now the case in which $1 - k > L$. Differentiating (17) with respect to x, we obtain

$$- \alpha \frac{\partial \theta_x^*}{\partial x} q \left[R \left(\theta_x^* + x \right) - (1-k) r_2 \right] + \alpha \int_{\theta_x^*}^{1-x} R d\theta - \alpha \frac{\partial^2 \theta_x^*}{\partial q \partial x} q \left[R \left(\theta_x^* + x \right) - (1-k) r_2 \right]$$
$$- \alpha \frac{\partial \theta_x^*}{\partial q} q \frac{\partial \theta_x^*}{\partial x} R - \alpha \frac{\partial \theta_x^*}{\partial q} q R.$$

Since $\frac{\partial \theta_x^*}{\partial x} = -1$ and $\frac{\partial^2 \theta_x^*}{\partial q \partial x} = \frac{\partial^2 \theta_x^*}{\partial x \partial q} = 0$, the expression above simplifies to

$$\alpha q \left[R \left(\theta_x^* + x \right) - (1-k) r_2 \right] + \alpha \int_{\theta_x^*}^{1-x} R d\theta > 0,$$

and the proposition follows. $\square$

Proof of Proposition 5: When $1 - k \leq L$, the relevant run threshold is $\underline{\theta}_x^P$, which corresponds to the solution to

$$R(\theta + x) - (1-k) r_2 = 0,$$

since when θ falls below $\underline{\theta}_x^P$ depositors expect to receive

$$q \frac{Rx}{1-k} + (1-q) \frac{Rx}{1-k} < 1,$$

and so prefer to run. Hence, $\underline{\theta}_x^P = \underline{\theta}_x$ holds.

When $1 - k > L$, the relevant run threshold is θ_x^{*P}. Following the same steps as in in the proof of Proposition 3, the threshold θ_x^{*P} is pinned down by a depositor's indifference condition, which corresponds to expression (18) in the proposition.

To complete the proof, we need to compute the effect of q, k and x on θ_x^{*P}. We do this by using the implicit function theorem. Denote as $f(x, q, k, \theta) = 0$ the indifference condition in (18). Thus,

$$\frac{d\theta_x^{*P}}{dq} = -\frac{\frac{\partial f(.)}{\partial q}}{\frac{\partial f(.)}{\partial \theta}}, \quad \frac{d\theta_x^{*P}}{dk} = -\frac{\frac{\partial f(.)}{\partial k}}{\frac{\partial f(.)}{\partial \theta}}, \quad \frac{d\theta_x^{*P}}{dx} = -\frac{\frac{\partial f(.)}{\partial x}}{\frac{\partial f(.)}{\partial \theta}}$$

The denominator

$$\frac{\partial f(.)}{\partial \theta}=\frac{\partial \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial \theta}q\left[r_2-\frac{Rx\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)\frac{(1-k)}{L}\right)}{\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)\right)(1-k)}\right]>0,$$

since $\widehat{n}_x\left(\theta_x^{*P}\right)=\frac{R\left(\theta_x^{*P}+x\right)-(1-k)r_2}{R\left(\theta_x^{*P}+x\right)\frac{(1-k)}{L}-(1-k)r_2}$ and so $\frac{\partial \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial \theta}=\frac{R-R\widehat{n}_x\left(\theta_x^{*P}\right)\frac{(1-k)}{L}}{R\left(\theta_x^{*P}+x\right)\frac{(1-k)}{L}-(1-k)r_2}=\frac{R\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)\frac{(1-k)}{L}\right)}{R\left(\theta_x^{*P}+x\right)\frac{(1-k)}{L}-(1-k)r_2}>0$. Hence, the signs of the effect of q,k and x on θ_x^{*P} are given by the opposite sign of the respective numerators. We have the following:

$$\frac{\partial f(.)}{\partial q}=\int_0^{\widehat{n}_x\left(\theta_x^{*P}\right)}r_2dn-\int_0^{\overline{n}}\frac{Rx\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn+\int_{\widehat{n}_x\left(\theta_x^{*P}\right)}^{\overline{n}}\frac{Rx\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn>0,$$

$$\begin{aligned}\frac{\partial f(.)}{\partial k}=&\frac{\partial \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial k}q\left[r_2-\frac{Rx\left(1-\widehat{n}_x\left(\theta^*\right)\frac{(1-k)}{L}\right)}{\left(1-\widehat{n}_x\left(\theta^*\right)\right)(1-k)}\right]+\int_{\widehat{n}_x\left(\theta_x^{*P}\right)}^{\overline{n}}q\frac{Rx}{(1-n)(1-k)^2}dn+\\&\int_0^{\overline{n}}(1-q)\frac{Rx}{(1-n)(1-k)^2}dn-\int_{\overline{n}}^1\frac{L}{(1-k)^2n}dn.\end{aligned}$$

The expression for $\frac{\partial f(.)}{\partial k}$ can be rearranged as

$$\begin{aligned}\frac{\partial f(.)}{\partial k}=&\frac{\partial \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial k}qr_2-\int_{\overline{n}}^1\frac{L}{(1-k)^2n}dn-\frac{\partial \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial k}q\frac{Rx\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)\frac{(1-k)}{L}\right)}{\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)\right)(1-k)}\\&+\int_{\widehat{n}_x\left(\theta_x^{*P}\right)}^{\overline{n}}q\frac{Rx}{(1-n)(1-k)^2}dn+\int_0^{\overline{n}}(1-q)\frac{Rx}{(1-n)(1-k)^2}dn,\end{aligned}\tag{39}$$

where $\frac{\partial \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial k}=\frac{R\left(\theta^{*P}+x\right)\frac{(1-k)}{L}-(1-k)r_2}{R\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)\frac{(1-k)}{L}\right)}>0$.

To establish the sign of (39), first notice that the first two terms sum up to a positive. This follows directly from the proof of Proposition 3, where we have shown that θ_x^* is decreasing in k. The derivative $\frac{\partial \theta_x^*}{\partial k}$ can be computed using the implicit function theorem from (36) as follows:

$$\frac{\partial \theta_x^*}{\partial k}=-\frac{\frac{\partial \widehat{n}_x(\theta^*)}{\partial k}qr_2-\int_{\overline{n}}^1\frac{L}{(1-k)^2n}dn}{\frac{\partial \widehat{n}_x(\theta^*)}{\partial \theta}qr_2}<0.$$

Given that $\frac{\partial \widehat{n}_x(\theta^*)}{\partial \theta}>0$, $\frac{\partial \theta_x^*}{\partial k}$ implies that $\frac{\partial \widehat{n}_x(\theta^*)}{\partial k}qr_2-\int_{\overline{n}}^1\frac{L}{(1-k)^2n}dn>0$. Since $\theta_x^*>\theta_x^{*P}$ and $\frac{\partial \widehat{n}_x(\theta^*)}{\partial k}$ is increasing in θ_x^*, it follows that when $\frac{\partial \widehat{n}_x(\theta^*)}{\partial k}qr_2-\int_{\overline{n}}^1\frac{L}{(1-k)^2n}dn>0$, also $\frac{\partial \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial k}qr_2-\int_{\overline{n}}^1\frac{L}{(1-k)^2n}dn>0$ holds. Hence, the sum of the first two terms in (39) is positive. A sufficient condition for $\frac{\partial f(.)}{\partial k}>0$ and so for $\frac{\partial \theta^{*P}}{\partial k}<0$ is that

$$\frac{\partial \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial k}q\frac{Rx\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)\frac{(1-k)}{L}\right)}{\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)\right)(1-k)}<\int_{\widehat{n}_x\left(\theta_x^{*P}\right)}^{\overline{n}}q\frac{Rx}{(1-n)(1-k)^2}dn,$$

that is

$$\frac{qRx}{L(1-k)}\left[\frac{\partial\widehat{n}_x\left(\theta_x^{*P}\right)}{\partial k}\frac{\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)\frac{(1-k)}{L}\right)}{\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)\right)(1-k)}-\frac{1}{1-k}\int_{\widehat{n}_x\left(\theta_x^{*P}\right)}^{\overline{n}}\frac{1}{(1-n)}dn\right]<0.$$

Substituting the expression for $\frac{\partial\widehat{n}_x(\theta^*)}{\partial k}$, we can express the sufficient condition simply as:

$$\frac{1}{1-k}\left[\frac{R\left(\theta_x^{*P}+x\right)\frac{(1-k)}{L}-(1-k)r_2}{R\ \left(1-\widehat{n}_x\left(\theta_x^{*P}\right)\right)}-\int_{\widehat{n}_x\left(\theta_x^{*P}\right)}^{\overline{n}}\frac{1}{(1-n)}dn\right]<0.$$

The inequality above holds because the integral $\int_{\widehat{n}_x\left(\theta_x^{*P}\right)}^{\overline{n}}\frac{1}{(1-n)}dn$ is increasing in n and is greater than $\frac{1}{1-\widehat{n}_x\left(\theta_x^{*P}\right)}$ and $\frac{R\left(\theta^{*P}+x\right)\frac{(1-k)}{L}-(1-k)r_2}{R}<1$.

Consider now the effect of x on θ_x^{*P}. We have the following:

$$\frac{\partial f(.)}{\partial x}=\frac{\partial\widehat{n}_x\left(\theta_x^{*P}\right)}{\partial x}q\left[r_2-\frac{Rx\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}\right]+\int_{\widehat{n}_x\left(\theta_x^{*P}\right)}^{\overline{n}}q\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn$$

$$+\int_0^{\overline{n}}(1-q)\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn>0,$$

and so the proposition follows. □

Proof of Proposition 6: As usual, we consider separately the case when $1-k\leq L$ and $1-k>L$. We start from the former. When $1-k\leq L$, the first order condition with respect to q is given by (57), which implies that the sign of $\frac{dq^{*P}}{dx}$ is equal to the sign of the expression in (37). As shown in the proof of Proposition 4, this is always positive.

Consider now the case when $1-k>L$. The first order condition with respect to q is given in (19). As q^{*P} is an interior solution, using the implicit function theorem, the sign of $\frac{dq^{*P}}{dx}$ is equal to the sign of the derivative of (19) with respect to x. Differentiating (19) with respect to x, we obtain

$$\begin{aligned}&-\alpha\frac{\partial\theta_x^{*P}}{\partial x}\left[R\left(\theta_x^{*P}+x\right)-(1-k)r_2\right]-\alpha\frac{\partial\theta_x^{*P}}{\partial q}\frac{\partial\theta_x^{*P}}{\partial x}qR\\&+\alpha\int_{\theta_x^{*P}}^{1-x}Rd\theta-\alpha\frac{\partial^2\theta_x^{*P}}{\partial q\partial x}q\left[R\left(\theta_x^{*P}+x\right)-(1-k)r_2\right]-\alpha\frac{\partial\theta_x^{*P}}{\partial q}qR,\end{aligned}\tag{40}$$

which can be further rearranged as follows:

$$-\alpha\left[\frac{\partial\theta_x^{*P}}{\partial x}+\frac{\partial^2\theta_x^{*P}}{\partial q\partial x}q\right]\left[R\left(\theta_x^{*P}+x\right)-(1-k)r_2\right]-\alpha\frac{\partial\theta_x^{*P}}{\partial q}\frac{\partial\theta_x^{*P}}{\partial x}qR+\alpha\int_{\theta_x^{*P}}^{1-x}Rd\theta-\alpha\frac{\partial\theta_x^{*P}}{\partial q}qR.\tag{41}$$

To establish the sign of the expression above, we need to compute $\frac{\partial^2\theta_x^{*P}}{\partial q\partial x}$. Recall that

$$\frac{\partial\theta_x^{*P}}{\partial x}=-1-\frac{q\int_{\widehat{n}_x\left(\theta_x^{*P}\right)}^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{q\frac{\partial\widehat{n}_x\left(\theta_x^{*P}\right)}{\partial\theta}\left[r_2-\frac{Rx\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)\frac{(1-k)}{L}\right)}{\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)\right)(1-k)}\right]}-\frac{\int_0^{\overline{n}}(1-q)\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{q\frac{\partial\widehat{n}_x\left(\theta_x^{*P}\right)}{\partial\theta}\left[r_2-\frac{Rx\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)\frac{(1-k)}{L}\right)}{\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)\right)(1-k)}\right]}<0,\tag{42}$$

and

$$\frac{\partial \theta^{*P}}{\partial q} = -\frac{1}{q}\frac{\int_0^{\widehat{n}_x\left(\theta_x^{*P}\right)} r_2 dn + \int_{\widehat{n}_x\left(\theta_x^{*P}\right)}^{\overline{n}} \frac{Rx\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)} dn - \int_0^{\overline{n}} \frac{Rx\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)} dn}{\frac{\partial \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial \theta}\left[r_2 - \frac{Rx\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)^{\frac{(1-k)}{L}}\right)}{(1-\widehat{n}_x(\theta_x^{*P}))(1-k)}\right]} < 0. \tag{43}$$

Since $\frac{\partial^2 \theta_x^{*P}}{\partial x \partial q} = \frac{\partial^2 \theta_x^{*P}}{\partial q \partial x}$, we can differentiate (43) with respect to x. Before doing this, in order to keep the notation compact, denote as Φ the denominator $\frac{\partial \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial \theta}\left[r_2 - \frac{Rx\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)^{\frac{(1-k)}{L}}\right)}{(1-\widehat{n}_x(\theta_x^{*P}))(1-k)}\right]$.

$$\begin{aligned}
\frac{\partial^2 \theta_x^{*P}}{\partial x \partial q} &= -\frac{1}{q}\frac{1}{\Phi}\left[\int_{\widehat{n}_x\left(\theta_x^{*P}\right)}^{\overline{n}} \frac{R\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)} dn + \quad \frac{\partial \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial \theta}\frac{\partial \theta_x^{*P}}{\partial x} + \frac{\partial \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial x}\right)\frac{\Phi}{\frac{\partial \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial \theta}}\right] \\
&+ \frac{1}{q}\frac{1}{\Phi}\int_0^{\overline{n}} \frac{R\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)} dn - \frac{1}{q}\frac{1}{\Phi}\frac{\partial \theta_x^{*P}}{\partial q}\frac{\partial \Phi}{\partial \theta}\frac{\partial \theta_x^{*P}}{\partial x} - \frac{1}{q}\frac{1}{\Phi}\frac{\partial \theta_x^{*P}}{\partial q}\left[\frac{\partial \Phi}{\partial \theta}\frac{\partial \theta_x^{*P}}{\partial x} + \frac{\partial \Phi}{\partial x}\right] \\
&= \frac{1}{q}\frac{1}{\Phi}\int_0^{\widehat{n}_x\left(\theta_x^{*P}\right)} \frac{R\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)} dn - \frac{1}{q}\frac{\partial \theta_x^{*P}}{\partial x} - \frac{1}{q} - \frac{1}{q}\frac{1}{\Phi}\frac{\partial \theta_x^{*P}}{\partial q}\left[\frac{\partial \Phi}{\partial \theta}\frac{\partial \theta_x^{*P}}{\partial x} + \frac{\partial \Phi}{\partial x}\right]
\end{aligned}$$

with $\frac{\partial \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial \theta} = \frac{\partial \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial x}$, $\frac{\partial \Phi}{\partial \theta} = \frac{\partial \widehat{n}_x^2\left(\theta_x^{*P}\right)}{\partial \theta^2}\left[r_2 - \frac{Rx\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)^{\frac{(1-k)}{L}}\right)}{(1-\widehat{n}_x(\theta_x^{*P}))(1-k)}\right] + \left(\frac{\partial \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial \theta}\right)^2 \frac{Rx\left(\frac{1-k}{L}-1\right)(2-\widehat{n}_x)}{(1-\widehat{n}_x)^2(1-k)}$, $\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial \theta} - \frac{\partial \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial \theta}\frac{R\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)^{\frac{(1-k)}{L}}\right)}{(1-\widehat{n}_x(\theta_x^{*P}))(1-k)}$ and $\frac{\partial^2 \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial \theta^2} = \frac{\partial^2 \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial \theta \partial x} = -\frac{2\frac{\partial \widehat{n}_x\left(\theta_x^{*P}\right)}{\partial \theta} R\frac{(1-k)}{L}}{R(\theta_x^{*P}+x)\frac{(1-k)}{L} - (1-k)r_2} < 0.$

Rearranging $\frac{\partial \theta_x^{*P}}{\partial x} = -1 + \frac{1}{\Phi}\int_0^{\widehat{n}_x\left(\theta_x^{*P}\right)} \frac{R\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)} dn - \frac{1}{q}\frac{1}{\Phi}\int_0^{\overline{n}} \frac{R\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)} dn$, we can rewrite the expression for $\frac{\partial^2 \theta_x^{*P}}{\partial x \partial q}$ as follows:

$$\begin{aligned}
\frac{\partial^2 \theta_x^{*P}}{\partial x \partial q} &= +\frac{1}{q}\frac{1}{\Phi}\int_0^{\widehat{n}_x\left(\theta_x^{*P}\right)} \frac{R\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)} dn - \frac{1}{q}\frac{1}{\Phi}\int_0^{\widehat{n}_x\left(\theta_x^{*P}\right)} \frac{R\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)} dn + \frac{1}{q^2}\frac{1}{\Phi}\int_0^{\overline{n}} \frac{R\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)} dn \\
&\quad - \frac{1}{q}\frac{1}{\Phi}\frac{\partial \theta_x^{*P}}{\partial q}\left(\frac{\partial \Phi}{\partial \theta}\frac{\partial \theta_x^{*P}}{\partial x} + \frac{\partial \Phi}{\partial x}\right) \\
&= \frac{1}{q}\left[\frac{1}{q}\frac{1}{\Phi}\int_0^{\overline{n}} \frac{R\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)} dn - \frac{1}{\Phi}\frac{\partial \theta_x^{*P}}{\partial q}\left(\frac{\partial \Phi}{\partial \theta}\frac{\partial \theta_x^{*P}}{\partial x} + \frac{\partial \Phi}{\partial x}\right)\right].
\end{aligned}$$

Hence, the expression in (41) can be written as

$$\begin{aligned}
&-\alpha\left[-1 + \frac{1}{\Phi}\int_0^{\widehat{n}_x\left(\theta_x^{*P}\right)} \frac{R\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)} dn - \frac{1}{\Phi}\frac{\partial \theta_x^{*P}}{\partial q}\left(\frac{\partial \Phi}{\partial \theta}\frac{\partial \theta_x^{*P}}{\partial x} + \frac{\partial \Phi}{\partial x}\right)\right]\left[R\left(\theta_x^{*P} + x\right) - (1-k) r_2\right] \\
&+ \alpha\int_{\theta_x^{*P}}^{1-x} R d\theta - \alpha\frac{\partial \theta_x^{*P}}{\partial q} qR\left[\frac{1}{\Phi}\int_0^{\widehat{n}_x\left(\theta_x^{*P}\right)} \frac{R\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)} dn - \frac{1}{q}\frac{1}{\Phi}\int_0^{\overline{n}} \frac{R\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)} dn\right].
\end{aligned} \tag{44}$$

First, given that $\overline{n} > \widehat{n}_x\left(\theta_x^{*P}\right)$ and $q < 1$, one can see that

$$\frac{1}{\Phi}\int_0^{\widehat{n}_x\left(\theta_x^{*P}\right)} \frac{R\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)}dn - \frac{1}{q}\frac{1}{\Phi}\int_0^{\overline{n}} \frac{R\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)}dn < 0,$$

which implies that the last term in (44) is negative since $\frac{\partial\theta_x^{*P}}{\partial q} < 0$.

Consider now the terms in the first bracket and denote it as Λ. We want to show that $\Lambda > 0$ that is

$$\Lambda = -1 + \frac{1}{\Phi}\int_0^{\widehat{n}_x\left(\theta_x^{*P}\right)} \frac{R\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)}dn - \frac{1}{\Phi}\frac{\partial\theta_x^{*P}}{\partial q}\left(\frac{\partial\Phi}{\partial\theta}\frac{\partial\theta_x^{*P}}{\partial x} + \frac{\partial\Phi}{\partial x}\right) > 0$$

Using $\frac{\partial\Phi}{\partial x} = \frac{\partial\Phi}{\partial\theta} - \frac{\partial\widehat{n}_x\left(\theta_x^{*P}\right)}{\partial\theta}\frac{R\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta_x^{*P}))(1-k)}$, we can rewrite

$$\frac{\partial\Phi}{\partial\theta}\frac{\partial\theta_x^{*P}}{\partial x} + \frac{\partial\Phi}{\partial x} = \frac{\partial\Phi}{\partial\theta}\left(\frac{\partial\theta_x^{*P}}{\partial x} + 1 - \frac{\partial\widehat{n}_x\left(\theta_x^{*P}\right)}{\partial\theta}\frac{R\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta_x^{*P}))(1-k)}\right),$$

so that

$$\Lambda = -1 + \frac{1}{\Phi}\int_0^{\widehat{n}_x\left(\theta_x^{*P}\right)} \frac{R\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)}dn - \frac{1}{\Phi}\frac{\partial\theta_x^{*P}}{\partial q}\frac{\partial\Phi}{\partial\theta}\left(\frac{\partial\theta_x^{*P}}{\partial x} + 1 - \frac{\partial\widehat{n}_x\left(\theta_x^{*P}\right)}{\partial\theta}\frac{R\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta_x^{*P}))(1-k)}\right).$$

When $x \to 0$, $\frac{\partial\Phi}{\partial\theta} = \frac{\partial\widehat{n}_x^2\left(\theta_x^{*P}\right)}{\partial\theta^2}r_2 < 0$ and $\frac{\partial\theta_x^{*P}}{\partial q}\frac{\partial\Phi}{\partial\theta} \quad \frac{\partial\theta_x^{*P}}{\partial x} + 1 - \frac{\partial\widehat{n}_x\left(\theta_x^{*P}\right)}{\partial\theta}\frac{R\left(1-\widehat{n}_x\left(\theta_x^{*P}\right)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta_x^{*P}))(1-k)}\Bigg) < 0$ since $\frac{\partial\theta_x^{*P}}{\partial x} < -1$.

Furthermore, the first two terms in Λ simplify to

$$-1 + \frac{1}{\Phi}\int_0^{\widehat{n}_x(\theta)} \frac{R\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)}dn = \frac{\int_0^{\widehat{n}_x(\theta)}\frac{R\theta_x^{*P}\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)}dn}{\frac{R\theta_x^{*P}\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)}}\frac{\left[R\theta_x^{*P}\frac{(1-k)}{L} - (1-k)r_2\right]}{r_2} - 1 > 0,$$

since $\frac{\int_0^{\widehat{n}_x(\theta)}\frac{R\theta_x^{*P}\left(1-n^{\frac{(1-k)}{L}}\right)}{(1-n)(1-k)}dn}{\frac{R\theta_x^{*P}\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)}} > 1$ and $\frac{R\theta_x^{*P}\frac{(1-k)}{L}}{r_2} - (1-k) = \frac{(1-k)}{L}\left[\frac{R\theta_x^{*P}}{r_2} - L\right] > 1$ given that $1-k > L$ and $R\theta_x^{*P} > (1-k)r_2$. It follows that $\Lambda > 0$ and overall that the first and last terms in (44) are negative, while the second one is positive.

When $x \to 0$, $\theta_x^{*P} \to \theta^*$. Given that $\theta^* \to \overline{\theta}$, which is arbitrarily close to 1, as $k \to 0$, it follows that the expression in (44) becomes negative as $k \to 0$ since $\theta_x^{*P} \to 1$. By continuity, it continues to be negative also for k small but strictly larger than 0. Similarly, given that when $1-k = L$ the entire expression is positive. It follows that in the range $k \in (0, 1-L)$, there exists a cutoff $\widehat{k}_x^P$, such that $\frac{dq^{*P}}{dx} < 0$ for $k < \widehat{k}_x^P$ and $\frac{dq^{*P}}{dx} > 0$ for $k > \widehat{k}_x^P$. Hence, the proposition follows. $\square$

Proof of Proposition 7: The proof proceeds in steps: First, we characterize depositors' withdrawal behavior. Then, we solve for the optimal q and characterize the effect of the introduction of the guarantees on the bank's monitoring choice. In doing so, we distinguish between the case in which the guaranteed amount is lost in bankruptcy and when it is protected from bankruptcy. We start with the former.

The characterization of depositors' withdrawal decision follows the same steps as in the proof of Propositions 3 and 5. Running is a dominant strategy when $\theta < \underline{\theta}_y$, which corresponds to the solution to

$$R\left[\theta + (1-\theta)\, y\right] - (1-k)\, r_2 = 0,$$

which gives

$$\underline{\theta}_y = \frac{\underline{\theta} - y}{1-y},$$

with $\underline{\theta} = \frac{(1-k)r_2}{R}$ corresponding to the run threshold when there are no guarantees, as given in (5).

When $1-k > L$, banks are exposed to panic runs. Following the same steps as in the previous sections, the condition pinning down θ_y^* is

$$\int_0^{\widehat{n}_y(\theta)} q r_2 dn = \pi_1,$$

where π_1 is given in (32) and $\widehat{n}_y(\theta)$ solves

$$R\left[\theta + (1-\theta)\, y\right]\left(1 - \frac{n\,(1-k)}{L}\right) - (1-n)\,(1-k)\, r_2 = 0.$$

$$\widehat{n}_y(\theta) = L \frac{r_2 - R\theta - kr_2 - Ry + Ry\theta}{(k-1)\,(R\theta - Lr_2 + Ry - Ry\theta)}.$$

After a few manipulations, we obtain the expression in the proposition,

$$\theta_y^* = \frac{\theta^* - y}{1-y},$$

where θ^* corresponds to the run threshold when there are no guarantees, as given in (6). As shown in the proof of Proposition 1, $\underline{\theta}_y$ and θ_y^* are the relevant run thresholds for banks with high capital (i.e., $1-k \leq L$) and low capital ($1-k > L$), respectively.

We now move on to the choice of q. When $1-k \leq L$, the bank solves the following problem:

$$\max_q \alpha q \int_0^{\underline{\theta}_y} R\left[\theta + (1-\theta)\, y\right]\left(1 - \frac{(1-k)}{L}\right) d\theta + \alpha q \int_{\underline{\theta}_y}^1 \left[R\left[\theta + (1-\theta)\, y\right] - (1-k)\, r_2\right] d\theta$$
$$+ (1-\alpha)\, q \int_1^2 \left[R - (1-k)\, r_2\right] d\theta - \frac{cq^2}{2},$$

while when $1-k>L$, the objective function is

$$\max_q \alpha q \int_{\theta_y^*}^1 \left[R\left[\theta+(1-\theta)y\right]-(1-k)r_2\right]d\theta+(1-\alpha)q\int_1^2\left[R-(1-k)r_2\right]d\theta-\frac{cq^2}{2}.$$

The first order condition for q is

$$\alpha\int_0^{\underline{\theta}_y} R\left[\theta+(1-\theta)y\right]\left(1-\frac{(1-k)}{L}\right)d\theta+\alpha\int_{\underline{\theta}_y}^1\left[R\left[\theta+(1-\theta)y\right]-(1-k)r_2\right]d\theta$$
$$+(1-\alpha)\int_1^2\left[R-(1-k)r_2\right]d\theta-cq=0,$$

when $1-k\leq L$, since $\frac{\partial\underline{\theta}_y}{\partial q}=0$ and

$$\alpha\int_{\theta_y^*}^1\left[R\left[\theta+(1-\theta)y\right]-(1-k)r_2\right]d\theta-\alpha\frac{\partial\theta_y^*}{\partial q}q\left[R\left[\theta_y^*+\left(1-\theta_y^*\right)y\right]-(1-k)r_2\right]$$
$$+(1-\alpha)\int_1^2\left[R-(1-k)r_2\right]d\theta-cq=0, \tag{45}$$

when $1-k>L$, with $\frac{\partial\theta_y^*}{\partial q}=\frac{1}{1-y}\frac{\partial\theta^*}{\partial q}<0$.

To compute the effect of y on the optimal q, we use the implicit function theorem. Thus, the sign of $\frac{dq_y}{dy}$ is equal to the sign of $\frac{\partial FOC_q}{\partial y}$. When $1-k\leq L$, $\frac{\partial FOC_q}{\partial y}$ is equal to:

$$\alpha\int_0^{\underline{\theta}_y}R(1-\theta)\left(1-\frac{(1-k)}{L}\right)d\theta+\alpha\int_{\underline{\theta}_y}^1R(1-\theta)\,d\theta+\alpha\frac{\partial\underline{\theta}_y}{\partial y}R\left[\underline{\theta}_y+\left(1-\underline{\theta}_y\right)y\right]\left(1-\frac{(1-k)}{L}\right).$$

The first two terms are positive, while the last one is negative since $\frac{\partial\underline{\theta}_y}{\partial y}=-\frac{1-\underline{\theta}_y}{1-y}$. When $1-k=L$, $\frac{\partial FOC_q}{\partial y}$ simplifies to $\alpha\int_{\underline{\theta}_y}^1R(1-\theta)\,d\theta>0$. As $k\to1$, then $\underline{\theta}_y\to0$ for any $y>\underline{\theta}$, and so $\frac{\partial\underline{\theta}_y}{\partial y}=0$, while for $y<\underline{\theta}_y$ and $y\to0$, the term $\underline{\theta}_y+\left(1-\underline{\theta}_y\right)y\to0$. It follows that $\frac{\partial FOC_q}{\partial y}>0$ for all $k\in(1-L,1)$, so that $\frac{d\underline{q}_y}{dy}>0$ holds.

Consider now the case when $1-k>L$: $\frac{\partial FOC_q}{\partial y}$ is given by

$$\alpha\int_{\theta_y^*}^1R(1-\theta)\,d\theta-\alpha\frac{\partial\theta_y^*}{\partial y}\left[R\left[\theta_y^*+\left(1-\theta_y^*\right)y\right]-(1-k)r_2\right]-\alpha\frac{\partial\theta_y^*}{\partial q}q\frac{\partial\theta_y^*}{\partial y}R(1-y)$$
$$-\alpha\frac{\partial\theta_y^*}{\partial q}qR\left(1-\theta_y^*\right)-\alpha\frac{\partial^2\theta_y^*}{\partial q\partial y}q\left[R\left[\theta_y^*+\left(1-\theta_y^*\right)y\right]-(1-k)r_2\right],$$

where $\frac{\partial^2\theta_y^*}{\partial q\partial y}=\frac{\partial^2\theta_y^*}{\partial y\partial q}=\frac{1}{1-y}\frac{\partial\theta_y^*}{\partial q}<0$. All terms in the expression above are positive except

$$-\frac{\partial\theta_y^*}{\partial q}q\frac{\partial\theta_y^*}{\partial y}R(1-y)<0.$$

Recall that $\frac{\partial\theta_y^*}{\partial y}=-\frac{1-\theta_y^*}{1-y}<0$. Then, we can write

$$-\frac{\partial\theta_y^*}{\partial q}q\frac{\partial\theta_y^*}{\partial y}R(1-y)-\frac{\partial\theta_y^*}{\partial q}qR\left(1-\theta_y^*\right)=-\frac{\partial\theta_y^*}{\partial q}qR\left[-\left(1-\theta_y^*\right)+\left(1-\theta_y^*\right)\right]=0,$$

and it follows that $\frac{dq_y^*}{dy} > 0$ when $1-k > L$.

We now move on to the case when the guarantee amount is protected from bankruptcy. The threshold for fundamental runs is still given by $\underline{\theta}_y$ as specified above. The threshold for panic runs θ_y^{*P}, instead, now solves:

$$\int_0^{\widehat{n}_y(\theta)} qr_2 dn + \int_{\widehat{n}_y(\theta)}^{\overline{n}} q \frac{R(1-\theta)y\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn + \int_0^{\overline{n}} (1-q) \frac{Ry\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn = \pi_1, \quad (46)$$

where π_1 and $\widehat{n}_y(\theta)$ are as above and $\overline{n}$ is still equal to $\frac{L}{1-k}$.

As we perform our analysis for the case in which $y \to 0$, the expression in (46) is increasing in θ and decreasing in n, so the usual derivations to characterize the panic run threshold θ_y^{*P} apply. Using the implicit function theorem, we can compute

$$\frac{\partial \theta_y^{*P}}{\partial q} = -\frac{\int_0^{\widehat{n}_y(\theta)} r_2 dn + \int_{\widehat{n}_y(\theta)}^{\overline{n}} \frac{R(1-\theta)y\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn - \int_0^{\overline{n}} \frac{Ry\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn}{\frac{\partial \widehat{n}_y(\theta)}{\partial \theta} q\left[r_2 - \frac{R(1-\theta)y\left(1-\widehat{n}_y(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_y(\theta))(1-k)}\right] - q\int_{\widehat{n}_y(\theta)}^{\overline{n}} \frac{Ry\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn} < 0,$$

and

$$\frac{\partial \theta_y^{*P}}{\partial y} = \frac{-\frac{\partial \widehat{n}_y(\theta)}{\partial y}\left[qr_2 - q\frac{R(1-\theta)y\left(1-\widehat{n}_y\frac{(1-k)}{L}\right)}{(1-\widehat{n}_y)(1-k)}\right] - q\int_{\widehat{n}_y(\theta)}^{\overline{n}} \frac{R(1-\theta)\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn - (1-q)\int_0^{\overline{n}} \frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn}{\frac{\partial \widehat{n}_y(\theta)}{\partial \theta} q\left[r_2 - \frac{R(1-\theta)y\left(1-\widehat{n}_y(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_y(\theta))(1-k)}\right] - q\int_{\widehat{n}_y(\theta)}^{\overline{n}} \frac{Ry\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn} < 0.$$

Starting from $\frac{\partial \theta_y^{*P}}{\partial y}$, we can compute $\frac{\partial^2 \theta_y^{*P}}{\partial q \partial y} = \frac{\partial^2 \theta_y^{*P}}{\partial y \partial q}$ as follows:

$$\begin{aligned}
\frac{\partial^2 \theta_y^{*P}}{\partial q \partial y} = &-\frac{\frac{\partial \widehat{n}_y(\theta)}{\partial y}\left[r_2 - \frac{R(1-\theta)y\left(1-\widehat{n}_y(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_y(\theta))(1-k)}\right] + \int_{\widehat{n}_y(\theta)}^{\overline{n}} \frac{R(1-\theta)\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn - \int_0^{\overline{n}} \frac{Ry\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn}{\frac{\partial \widehat{n}_y(\theta)}{\partial \theta} q\left[r_2 - \frac{R(1-\theta)y\left(1-\widehat{n}_y(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_y(\theta))(1-k)}\right] - q\int_{\widehat{n}_y(\theta)}^{\overline{n}} \frac{Ry\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn} \\
&- \frac{\partial \theta_y^{*P}}{\partial y} \frac{\frac{\partial \widehat{n}_y(\theta)}{\partial \theta}\left[r_2 - \frac{R(1-\theta)y\left(1-\widehat{n}_y(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_y(\theta))(1-k)}\right] - \int_{\widehat{n}_y(\theta)}^{\overline{n}} \frac{Ry\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn}{\frac{\partial \widehat{n}_y(\theta)}{\partial \theta} q\left[r_2 - \frac{R(1-\theta)y\left(1-\widehat{n}_y(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_y(\theta))(1-k)}\right] - q\int_{\widehat{n}_y(\theta)}^{\overline{n}} \frac{Ry\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn} > 0.
\end{aligned}$$

When $y = 0$, the expression above simplifies to

$$\begin{aligned}
\frac{\partial^2 \theta_y^{*P}}{\partial q \partial y} = &-\frac{1}{\frac{\partial \widehat{n}_y(\theta)}{\partial \theta} qr_2}\left[\frac{\partial \widehat{n}_y(\theta)}{\partial y} r_2 + \int_{\widehat{n}_y(\theta)}^{\overline{n}} \frac{R(1-\theta)\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn\right] \\
&- \frac{\partial \theta_y^{*P}}{\partial y}\frac{1}{q},
\end{aligned}$$

from which we can see that $\frac{\partial^2\theta_y^{*P}}{\partial q\partial y} > \frac{\partial\theta_y^{*P}}{\partial y}$ when $y = 0$.

The FOC_q is still given by (45), so that the expression for $\frac{\partial FOC_q}{\partial y}$ when $y = 0$ is given by

$$\alpha \int_{\theta_y^{*P}}^{1} R(1-\theta)\,d\theta - \alpha\frac{\partial\theta_y^{*P}}{\partial y}\left[R\theta_y^{*P} - (1-k)\,r_2\right]$$
$$-\alpha\frac{\partial\theta_y^{*P}}{\partial q}q\frac{\partial\theta_y^{*P}}{\partial y}R - \alpha\frac{\partial^2\theta_y^{*P}}{\partial q\partial y}q\left[R\theta_y^{*P} - (1-k)\,r_2\right] - \alpha\frac{\partial\theta_y^{*P}}{\partial q}qR\left(1-\theta_y^{*P}\right).$$

Again, all terms are positive except $-\alpha\frac{\partial\theta_y^{*P}}{\partial q}q\frac{\partial\theta_y^{*P}}{\partial y}R < 0$. When $1-k = L$, we know that $\frac{\partial FOC_q}{\partial y} > 0$ and so $\frac{dq_y^{*P}}{dy} > 0$. Recall that $\theta^* \to \overline{\theta}$, which is arbitrarily close to 1, as $k \to 0$. Hence, we can extend the argument to the case when $y = 0$, so that $\theta_y^{*P} \to \theta^*$ and conclude that when $k \to 0$, $\theta_y^{*P} \to \theta^* \to \overline{\theta}$, which is arbitrarily close to 1. Making use of this assumption, the expression for $\frac{\partial FOC_q}{\partial y}$ evaluated at $y = 0$ simplifies to

$$-\alpha\frac{\partial\theta_y^{*P}}{\partial y}\left[R - (1-k)\,r_2\right] - \alpha\frac{\partial^2\theta_y^{*P}}{\partial q\partial y}q\left[R - (1-k)\,r_2\right] - \alpha\frac{\partial\theta_y^{*P}}{\partial q}q\frac{\partial\theta_y^{*P}}{\partial y}R.$$

Since $\frac{\partial^2\theta_y^{*P}}{\partial q\partial y} > \frac{\partial\theta_y^{*P}}{\partial y}$ when $y = 0$, the expression above is negative, which implies that $\frac{dq_y^{*P}}{dy} < 0$. Using the same argument as in the proof of Proposition 6, we can establish that there exists a cutoff $\widehat{k}_y^P \in (0, 1-L)$ such that $\frac{dq_y^{*P}}{dy} < 0$ for $k < \widehat{k}_y^P$ and $\frac{dq_y^{*P}}{dy} > 0$ for $k > \widehat{k}_y^P$, so that the proposition follows. $\square$

Proof of Proposition 8: Given the expressions for GD_x and GD_y in (21) and (22) and evaluating (22) at $y = \underline{y} = \frac{x}{1-\max(\underline{\theta}-x,0)}$, the expression that determines which guarantee scheme is more costly is

$$Rx - \frac{Rx^2}{2} - Rx\left(\underline{\theta} - x\right)\frac{1-k}{L} \gtrless \frac{Rx}{2\left(1-\max\left(\underline{\theta}-x,0\right)\right)} - \frac{Rx}{\left(1-\max\left(\underline{\theta}-x,0\right)\right)}\left(\underline{\theta}-x\right)\frac{1-k}{L}$$
$$+\frac{Rx}{2\left(1-\max\left(\underline{\theta}-x,0\right)\right)}\frac{1-k}{L}\left(\underline{\theta}-x\right)^2,$$

which can be simplified as

$$\frac{Rx}{2}\left[(2-x) - \frac{1}{\left(1-\max\left(\underline{\theta}-x,0\right)\right)} - \left(\underline{\theta}-x\right)\frac{1-k}{L}\left(2 - \frac{2-\left(\underline{\theta}-x\right)}{\left(1-\max\left(\underline{\theta}-x,0\right)\right)}\right)\right] \gtrless 0. \qquad (47)$$

When (47) equals zero, the two guarantee schemes are equally costly. Note that this is the case for $x = 0$ and $x = 1$. When $x = 1$, $\max\left(\underline{\theta}-x,0\right) = 0$ and $\underline{\theta}_x = \underline{\theta} - x = 0$, so that the expression above simplifies to $\frac{R}{2}\left[1-1\right] = 0$.

We need now to check whether $GD_x \gtrless GD_y$ for any $x \in (0,1)$. Differentiate (47) with respect to x:

$$\frac{R}{2L\left(x\theta-1\right)^2}\left(L - x^2\underline{\theta}^3 + 2x^3\underline{\theta}^2 + 2x\underline{\theta}^2 - 3x^2\underline{\theta} - 2Lx + 2Lx^2\underline{\theta}^2 - 2Lx^3\underline{\theta}^2\right. \tag{48}$$
$$\left.+kx^2\underline{\theta}^3 - 2kx^3\underline{\theta}^2 - 4Lx\underline{\theta} + 4Lx^2\underline{\theta} - 2kx\underline{\theta}^2 + 3kx^2\underline{\theta}\right).$$

Evaluating (48) at $x = 0$ gives $\frac{L}{L} = 1 > 0$, so that, while the difference in the two loan guarantee schemes is zero at $x = 0$, it becomes positive as soon as x becomes positive.

We now show that the difference in (47) is concave everywhere, which implies that $GD_x > GD_y$ for any $x \in (0,1)$. Start by differentiating (48) with respect to x again to obtain

$$\frac{2}{L\left(x\underline{\theta}-1\right)^3}\left(L-(1-k)\right)\left(1+\underline{\theta}+3x^2\underline{\theta}^2 - x^3\underline{\theta}^3 - 3x\underline{\theta}\right) + (1-k)\,. \tag{49}$$

Since $\left(x\underline{\theta}-1\right)^3 < 0$, to show that $GD_x - GD_y$ is concave for any $x \in (0,1)$, we need to show that the expression in parentheses is positive.

For $x = 0$, the expression is clearly positive, meaning that the difference $GD_x - GD_y$ is concave around $x = 0$. A sufficient condition for the expression to be positive for any $x \in (0,1)$ is

$$1+\underline{\theta}+3x^2\underline{\theta}^2 - x^3\underline{\theta}^3 - 3x\underline{\theta} > 0.$$

This is equivalent to showing that

$$\frac{1+\underline{\theta}}{x\underline{\theta}} > -\left(3x\underline{\theta} - x^2\underline{\theta}^2 - 3\right). \tag{50}$$

Rewrite the LHS in (50) as $\frac{1}{x\underline{\theta}} + \frac{1}{x}$. From this, we can see that for any x, the value of the LHS is minimal at $\underline{\theta} = 1$, and equal to $\frac{2}{x}$.

Consider now the RHS in (50). Differentiating it with respect to $\underline{\theta}$ gives:

$$2x^2\underline{\theta} - 3x$$

This derivative is positive if $2x^2\underline{\theta} - 3x > 0 \Leftrightarrow 2x\underline{\theta} > 3 \Leftrightarrow x\underline{\theta} > \frac{3}{2}$, which can never happen since both x and $\underline{\theta}$ are less than 1. Hence, the RHS must be strictly decreasing in $\underline{\theta}$, and is maximized at $\underline{\theta} = 0$. For $\underline{\theta} = 0$, the RHS equals 3. The same thing is true for x: the RHS is decreasing in x, so the maximum value the RHS can take is 3, which occurs for either $x = 0$ or $\underline{\theta} = 0$.

Now consider the LHS. The lowest value it can take, as a function of x, is $\frac{2}{x}$. For this expression to become smaller than 3, i.e., the largest the RHS can be, we need $x > \frac{2}{3}$. Note now that x can only be greater than $\frac{2}{3}$ if $\underline{\theta}$ is also greater than $\frac{2}{3}$. Since the RHS is decreasing in x and $\underline{\theta}$, the most the RHS can be if $\frac{2}{3} \leq x \leq \underline{\theta}$ is

$$\left(\frac{2}{3}\right)^2\left(\frac{2}{3}\right)^2 - 3\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + 3 = 1.8642$$

which is less than the LHS.

Fix now $x = 1$. The lowest value that the LHS can take when $x = 1$ is 2. This is bigger than the value that the RHS takes when $x = \frac{2}{3}$. Thus, since both the LHS and the RHS are monotonically decreasing in x, it follows that the LHS is greater than the RHS for any $x > 0$. This implies, in turn, that the difference $GD_x - GD_y$ is concave for any $x \in (0, 1)$ and so it is always positive as stated in the proposition.

To complete the proof we need to determine the effect of the two guarantees schemes on q. To do so, we compare FOC_q under GS_x and GS_y. The former is equal to

$$\alpha \int_0^{\underline{\theta}_x} R\left(\theta + x\right)\left(1 - \frac{(1-k)}{L}\right) d\theta + \alpha \int_{\underline{\theta}_x}^{1-x} \left[R\left(\theta + x\right) - (1-k)\, r_2\right] d\theta$$
$$+ \alpha \int_{1-x}^{1} \left[R - (1-k)\, r_2\right] d\theta + (1-\alpha) \int_1^2 \left[R - (1-k)\, r_2\right] d\theta - cq = 0, \tag{51}$$

while the latter is equal to

$$\alpha \int_0^{\underline{\theta}_y} R\left(\theta + y - \theta y\right)\left(1 - \frac{(1-k)}{L}\right) d\theta + \alpha \int_{\underline{\theta}_y}^{1} \left[R\left(\theta + y - \theta y\right) - (1-k)\, r_2\right] d\theta$$
$$+ (1-\alpha) \int_1^2 \left[R - (1-k)\, r_2\right] d\theta - cq = 0, \tag{52}$$

since under GS_y the bank accrues a (per unit return) on the non-liquidated units equal to $R\theta + R\left(1-\theta\right) y = R\left(\theta + y - \theta y\right)$ and $\frac{\partial \underline{\theta}_y}{\partial q} = \frac{\partial \underline{\theta}_x}{\partial q} = 0$.

We now compare (51) and (52) evaluated at $y = \underline{y}$ so that $\underline{\theta}_y = \underline{\theta}_x$. Given that q_x and q_y are interior solutions, for $q_x > q_y$, it must be that

$$\alpha \int_0^{\underline{\theta}_x} R\left(\theta + x\right)\left(1 - \frac{(1-k)}{L}\right) d\theta + \alpha \int_{\underline{\theta}_x}^{1-x} \left[R\left(\theta + x\right) - (1-k)\, r_2\right] d\theta + \alpha \int_{1-x}^{1} \left[R - (1-k)\, r_2\right] d\theta$$
$$-\alpha \int_0^{\underline{\theta}_y} R\left(\theta + y - \theta y\right)\left(1 - \frac{(1-k)}{L}\right) d\theta - \alpha \int_{\underline{\theta}_y}^{1} \left[R\left(\theta + y - \theta y\right) - (1-k)\, r_2\right] d\theta \Bigg|_{y=\underline{y}} > 0.$$

After a few manipulations, we can rearrange the expression on the LHS of the inequality above as follows:

$$\alpha \int_0^{\underline{\theta}_x} R\left(x - \frac{x}{1+x-\underline{\theta}} + \theta\frac{x}{1+x-\underline{\theta}}\right)\left(1 - \frac{(1-k)}{L}\right) d\theta + \alpha \int_{\underline{\theta}_x}^{1-x} R\left(x - \frac{x}{1+x-\underline{\theta}} + \theta\frac{x}{1+x-\underline{\theta}}\right) d\theta$$
$$+ \alpha \int_{1-x}^{1} R\left(1 - \frac{x}{1+x-\underline{\theta}}\right)(1-\theta)\, d\theta$$
$$= \alpha \int_0^{\underline{\theta}_x} \frac{Rx^2}{1+x-\underline{\theta}}\left(1 - \frac{(1-k)}{L}\right) d\theta + \alpha \int_{\underline{\theta}_x}^{1-x} \frac{Rx^2}{1+x-\underline{\theta}} d\theta + \alpha \int_{1-x}^{1} \frac{R}{1+x-\underline{\theta}}\,(1-\underline{\theta})\,(1-\theta)\, d\theta > 0.$$

Hence, since $FOC_{q_x} > FOC_{q_y}$, it follows that $q_x > q_y$, as desired. □

Proof of Lemma 1: Substituting the expressions for θ_L^B and θ_L^{SP} from (23) and (24), respectively, it is easy to see that for any $0 \leq k < 1$ and $0 < q < 1$, $\theta_L^B < \theta_L^{SP}$ holds as

$$\frac{L - (1-q)(1-k)r_2}{qR} < \frac{L}{qR}.$$

The rest of the Lemma follows since θ_L^B increases with k, while θ_L^{SP} does not depend on k. □

Proof of Lemma 2: Comparing θ_B^L with $\underline{\theta}$, we have that

$$\theta_L^B < \underline{\theta} \iff r_2(1-k) > L.$$

Given that the LHS in the inequality above is decreasing in k and $\theta_L^B < \underline{\theta}$ when $k = 1 - L$ and $\theta_L^B > \underline{\theta}$ when $k = 1$, there exists a cutoff value $\overline{k}_L \in (1-L, 1)$ solving $\theta_L^B = \underline{\theta}$. Hence, the lemma follows. □

Proof of Proposition 9: When $1 - k \leq L$, the bank is exposed to fundamental runs only. The introduction of the loan guarantee reduces $\underline{\theta}_x$ and θ_{Lx}^B, while it does not affect the planner's threshold θ_L^{SP}. Hence, $\theta_L^{SP} - \max\left\{\theta_{Lx}^B, \underline{\theta}_x\right\}$ strictly decreases with x.

When $1 - k > L$, the bank is exposed to panic runs and the run threshold θ_x^* strictly decreases with x and k. Since $\underline{\theta}_x < \underline{\theta}$ and they are both decreasing in k, $\theta_L^{SP} = \underline{\theta}_x$ when $k = k_{L_x}^{SP} \equiv 1 - \frac{L}{qr_2} - \frac{x}{r_2} < 1 - L$. Hence, since $\theta_x^* > \underline{\theta}_x$, there exists a cutoff value $0 < \widetilde{k}_L < k_{L_x}^{SP} < 1 - L$ such that $\theta_L^{SP} \leq \theta_x^*$ when $k \leq \widetilde{k}_L$ and $\theta_L^{SP} > \theta_x^*$ when $k > \widetilde{k}_L$. The cutoff $\widetilde{k}_L$ solves $\theta_L^{SP} = \theta_x^*$ and the proposition follows. □

Proof of Lemma 3: Denote as k_L^{SP} the cutoff value of capital for which $\underline{\theta} = \theta_L^{SP}$. This is equal to

$$k_L^{SP} = 1 - \frac{L}{qr_2} \geq 1 - L,$$

for any $qr_2 \geq 1$. Given that $\frac{\partial \underline{\theta}}{\partial k} < 0$, while $\frac{\partial \theta_L^{SP}}{\partial k} = 0$, it follows that $\underline{\theta} > \theta_L^{SP}$ for $k < k_L^{SP}$ and $\underline{\theta} \leq \theta_L^{SP}$ for $k \geq k_L^{SP}$. From Proposition 2, we know that $qr_2 = 1$ when $1 - k \leq L$. Hence, it follows that $k_L^{SP} = 1 - L$ and, in turn, $\underline{\theta} \leq \theta_L^{SP}$ when $k \leq 1 - L$, while $\theta^* > \underline{\theta} > \theta_L^{SP}$ when $k > 1 - L$. Using the result from Lemma 2 that $\theta_L^B > \theta^R$ for $k > \overline{k}_L$ and $\theta_L^B \leq \theta^R$ for $k \leq \overline{k}_L$, we obtain the result in the lemma. □

Proof of Proposition 10: Since we are allowing the bank to choose to liquidate the project early if it finds it profitable to do so, we must first characterize the optimal degree of underwriting effort. This is obtained by maximizing bank profits with respect to q, and is given by

$$\max_q \alpha \int_0^{\theta_{Lx}^B} [L - (1-k)\, r_2]\, d\theta + \alpha \int_{\theta_{Lx}^B}^{1-x} q\, [R\,(\theta + x) - (1-k) r_2)]\, d\theta + \alpha \int_{1-x}^{1} q\, [R - (1-k)\, r_2]\, d\theta$$
$$+ (1-\alpha) \int_1^2 q\, [R - (1-k)\, r_2]\, d\theta - \frac{cq^2}{2},$$

which implies that q_{Lx}^B is the solution to

$$\alpha \int_{\theta_{Lx}^B}^{1-x} [R\,(\theta + x) - (1-k) r_2)]\, d\theta + \alpha \int_{1-x}^{1} [R - (1-k)\, r_2]\, d\theta + (1-\alpha) \int_1^2 [R - (1-k)\, r_2]\, d\theta = cq.$$

We can now calculate the change in total output resulting from an increase in x, which is given by

$$\begin{aligned} \frac{dTO_x}{dx} &= \alpha \frac{\partial \theta_{Lx}^B}{\partial x} \left(L - q_{Lx}^B R \theta_{Lx}^B \right) - \alpha q_{Lx}^B R\,(1-x) + \alpha q_{Lx}^B R\,(1-x) \\ &\quad + \frac{dq_{Lx}^B}{dx} \quad \alpha \int_{\theta_{Lx}^B}^{1-x} R\theta d\theta + \alpha \int_{1-x}^{1} R\theta d\theta + (1-\alpha) \int_1^2 R\theta d\theta - c q_{Lx}^B \Bigg), \end{aligned}$$

which simplifies to

$$\frac{dTO_x}{dx} = \alpha \frac{\partial \theta_{Lx}^B}{\partial x} \left(L - q_{Lx}^B R \theta_{Lx}^B \right) + \frac{dq_{Lx}^B}{dx} \quad \alpha \int_{\theta_{Lx}^B}^{1-x} R\theta d\theta + \alpha \int_{1-x}^{1} R\theta d\theta + (1-\alpha) \int_1^2 R\theta d\theta - c q_{Lx}^B \Bigg).$$

Recall that $\frac{\partial \theta_{Lx}^B}{\partial x} = -1$ and $\theta_{Lx}^B = \frac{L-(1-q_{Lx}^B)(1-k)r_2}{q_{Lx}^B R} - x$. Then, the expression above can be rearranged as

$$\begin{aligned} \frac{dTO_x}{dx} &= -\alpha \left(L - q_{Lx}^B R \left(\frac{L - (1-q_{Lx}^B)(1-k) r_2}{q_{Lx}^B R} - x \right) \right) \\ &\quad + \frac{dq_{Lx}^B}{dx} \left[\alpha \int_{\theta_{Lx}^B}^{1-x} R\theta d\theta + \alpha \int_{1-x}^{1} R\theta d\theta + (1-\alpha) \int_1^2 R\theta d\theta - c q_{Lx}^B \right] \\ &= -\alpha \left[(1-q_{Lx}^B)(1-k) r_2 + q_{Lx}^B R x \right] + \frac{dq_{Lx}^B}{dx} \left[\alpha \int_{\theta_{Lx}^B}^{1-x} R\theta d\theta + \alpha \int_{1-x}^{1} R\theta d\theta + (1-\alpha) \int_1^2 R\theta d\theta - c q_{Lx}^B \right]. \end{aligned} \tag{53}$$

At $k = 1$ this is

$$\frac{dTO_x}{dx} = -\alpha q_{Lx}^B R x + \frac{dq_{Lx}^B}{dx} \left[\alpha \int_{\theta_{Lx}^B}^{1-x} R\theta d\theta + \alpha \int_{1-x}^{1} R\theta d\theta + (1-\alpha) \int_1^2 R\theta d\theta - c q_{Lx}^B \right],$$

which is positive when $x \to 0$ since $\frac{dq^B_{Lx}}{dx} > 0$ from the expression above for q^B_{Lx}:

$$\frac{dq^B_{Lx}}{dx} = -\frac{-\alpha\frac{\partial\theta^B_{Lx}}{\partial x}\left[R\left(\theta^B_{Lx}+x\right)-(1-k)\,r_2\right]+\alpha\int_{\theta^B_{Lx}}^{1-x}Rd\theta}{-\alpha\frac{\partial\theta^B_{Lx}}{\partial q}\left[R\left(\theta^B_{Lx}+x\right)-(1-k)\,r_2\right]-c} > 0.$$

Hence, by continuity (53) is also positive for k close to but strictly less than 1.

Consider now the other extreme case when $k = \overline{k}_L$, which solves $1-k = \frac{L}{r_2}$. In this case,

$$\left.\frac{dq^B_{Lx}}{dx}\right|_{x=0,k=\overline{k}_L} = \frac{\alpha\int_{\frac{L}{R}}^{1}Rd\theta}{c} = \alpha\frac{(R-L)}{c},$$

and the expression for $\frac{dTO_x}{dx}$ becomes

$$\frac{dTO_x}{dx} = -\alpha\left((1-q^B_{Lx})L+q^B_{Lx}Rx\right)+\frac{dq^B_{Lx}}{dx}\left[\alpha\int_{\frac{L}{R}}^{1-x}R\theta d\theta+\alpha\int_{1-x}^{1}R\theta d\theta+(1-\alpha)\int_{1}^{2}R\theta d\theta-cq^B_{Lx}\right].$$

As $x \to 0$, this converges to

$$\frac{dTO_x}{dx} = -\alpha(1-q^B_{Lx})L+\alpha\frac{(R-L)}{c}\quad\alpha\int_{\frac{L}{R}}^{1}R\theta d\theta+(1-\alpha)\int_{1}^{2}R\theta d\theta-cq^B_{Lx}\Bigg).$$

Since

$$q^B_{Lx}\big|_{x=0,k=\overline{k}_L} = \frac{\alpha\left(\frac{R}{2}+\frac{L^2}{2R}-L\right)+(1-\alpha)(R-L)}{c}.$$

we can write

$$\begin{aligned}\frac{dTO_x}{dx} &= -\alpha\left(1-\frac{\alpha\left(\frac{R}{2}+\frac{L^2}{2R}-L\right)+(1-\alpha)(R-L)}{c}\right)L\\ &+\alpha\frac{(R-L)}{c}\quad\alpha\int_{\frac{L}{R}}^{1}R\theta d\theta+(1-\alpha)\int_{1}^{2}R\theta d\theta-\left(\alpha\left(\frac{R}{2}+\frac{L^2}{2R}-L\right)+(1-\alpha)(R-L)\right)\Bigg)\\ &= \frac{1}{2Rc}\alpha\left(-4L^2R+3LR^2+3L^3\alpha-R^3\alpha+R^3-2L^2R\alpha-2LRc\right).\end{aligned}$$

From this, we obtain that for

$$c < \frac{R^3(1-\alpha)+3LR^2+3L^3\alpha-2L^2R(2+\alpha)}{2LR},$$

we have that $\frac{dTO_x}{dx} > 0$. Hence, by continuity, $\frac{dTO_x}{dx} > 0$ for k larger but close to $\overline{k}_L$.

We now move on to show that $\frac{dTO_x}{dx} > 0$ is also positive in the range $k \in (\overline{k}_L, 1)$. Evaluating (53) at $x \to 0$ gives

$$\frac{dTO_x}{dx} = -\alpha(1-q^B_{Lx})(1-k)r_2+\frac{dq^B_{Lx}}{dx}\left[\alpha\int_{\theta^B_{Lx}}^{1}R\theta d\theta+(1-\alpha)\int_{1}^{2}R\theta d\theta-cq^B_{Lx}\right].$$

As c decreases, q^B_{Lx} increases and can come arbitrarily close to 1. Suppose c is sufficiently small that $q^B_{Lx} = 1-\delta$, for $\delta > 0$ but small. This makes the first term arbitrarily close to zero, of order $-O(\delta)$:

$$\frac{dTO_x}{dx} = -O(\delta) + \frac{dq^B_{Lx}}{dx}\left[\alpha\int_{\theta^B_{Lx}}^1 R\theta d\theta + (1-\alpha)\int_1^2 R\theta d\theta - c(1-\delta)\right].$$

The term inside the brackets is strictly positive. Therefore, $\frac{dTO_x}{dx}$ will be positive if $\frac{dq^B_{Lx}}{dx}$ remains bounded away from zero for q^B_{Lx} close to 1 but not strictly equal to 1. Recall that $\frac{dq^B_{Lx}}{dx}$ is given by

$$\frac{dq^B_{Lx}}{dx} = -\frac{-\alpha\frac{\partial\theta^B_{Lx}}{\partial x}\left[R\left(\theta^B_{Lx}+x\right)-(1-k)r_2\right] + \alpha\int_{\theta^B_{Lx}}^{1-x} Rd\theta}{-\alpha\frac{\partial\theta^B_{Lx}}{\partial q}\left[R\left(\theta^B_{Lx}+x\right)-(1-k)r_2\right] - c} > 0.$$

As $x \to 0$, this becomes

$$\frac{dq^B_{Lx}}{dx} = -\frac{-\alpha\frac{\partial\theta^B_{Lx}}{\partial x}\left[R\theta^B_{Lx}-(1-k)r_2\right] + \alpha\int_{\theta^B_{Lx}}^{1} Rd\theta}{-\alpha\frac{\partial\theta^B_{Lx}}{\partial q}\left[R\theta^B_{Lx}-(1-k)r_2\right] - c} > 0,$$

with $\frac{\partial\theta^B_{Lx}}{\partial x} = -1$. Therefore, even for c small, $\frac{dq^B_{Lx}}{dx}$ remains strictly positive, as long as $q^B_{Lx} < 1$ and that the second order condition continues to be satisfied, meaning that the denominator, $-\alpha\frac{\partial\theta^B_{Lx}}{\partial q}\left[R\theta^B_{Lx}-(1-k)r_2\right] - c$, remains negative.

Hence, for small enough c, $\frac{dTO_x}{dx} > 0$ for any $k \in \left(\overline{k}_L, 1\right)$. Since we know it is positive for $k=1$ and $k = \overline{k}_L$, this establishes that $\frac{dTO_x}{dx} > 0$ for all k in $\left[\overline{k}_L, 1\right]$. $\square$

Proof of Proposition 11: When $1-k \le L$, the deposit rate r_2 is pinned down from depositors' participation constraint, (8), with the small modification to adjust the limits of integration, θ^R, to be $\theta^R = \underline{\theta} - x$. Relative to the case of $x = 0$, the introduction of a loan guarantee introduces slack in depositors' participation constraint. Keeping its underwriting effort q constant, the bank can now reduce r_2 and still satisfy depositors' participation constraint. Together with the direct effect of the guarantee, the reduction in r_2 would lead to a further increase in q. But the consequent anticipated increase in q would again make depositors' participation constraint slack, allowing for a yet greater reduction in r_2, etc. Hence, r_2 decreases in equilibrium as a result of the introduction of the loan guarantee x.

When instead $1-k > L$, r_2 is pinned down either again by (8), or by the bank's first order condition, where the run threshold is given by $\theta^*_x = \theta^* - x$. In the case where $1-k > L$, bank profits are given by

$$\max_q \alpha\int_{\theta^*_x}^{1-x} q\left[R(\theta+x)-(1-k)r_2\right]d\theta + \alpha\int_{1-x}^{1} q\left[R-(1-k)r_2\right]d\theta + (1-\alpha)\int_1^2 q\left[R-(1-k)r_2\right]d\theta - \frac{cq^2}{2}.$$

The derivative with respect to r_2 is

$$- \alpha \frac{\partial \theta_x^*}{\partial r_2} q \left[R \left(\theta_x^* + x \right) - (1-k) r_2 \right] - \alpha \int_{\theta_x^*}^{1-x} q (1-k) \, d\theta - \alpha \int_{1-x}^{1} q (1-k) \, d\theta - (1-\alpha) \int_{1}^{2} q (1-k) \, d\theta$$

$$= -\alpha \frac{\partial \theta_x^*}{\partial r_2} q \left[R \left(\theta_x^* + x \right) - (1-k) r_2 \right] - \alpha \int_{\theta_x^*}^{1} q (1-k) \, d\theta - (1-\alpha) \int_{1}^{2} q (1-k) \, d\theta$$

Since $\theta_x^* = \theta^* - x$, the expression above can be further rearranged as follows:

$$-\alpha \frac{\partial \theta^*}{\partial r_2} q \left[R\theta^* - (1-k) r_2 \right] - \alpha \int_{\theta^* - x}^{1} q (1-k) \, d\theta - (1-\alpha) \int_{1}^{2} q (1-k) \, d\theta$$

The first term, which is positive, is exactly the same as for the case where $x = 0$, given by (12), while the second term, which is negative, is larger because of the larger region of integration. Hence, the bank should respond to the introduction of the guarantees by reducing r_2 and the proposition follows. $\square$

Proof of Proposition 12: Since $\delta < 1$, the threshold for fundamental runs is the same as in the case without guarantees. This is due to the fact that when the bank is insolvent depositors receive $\delta < 1$, but this is not enough to convince them not to run. Hence, for highly capitalized banks, when $1 - k \leq L$, $\underline{\theta}_\delta$ is still given by (5).

Applying the same arguments as in the proof of Proposition 1, for banks with $1 - k > L$, the relevant crisis threshold θ_δ^* corresponds to the solution to

$$\int_0^{\widehat{n}(\theta)} q r_2 dn + \int_{\widehat{n}(\theta)}^{1} q \delta dn + \int_0^1 (1-q) \, \delta dn = \pi_1,$$

or, equivalently,

$$q \int_0^{\widehat{n}(\theta)} (r_2 - \delta) \, dn + \int_0^1 \delta dn = \pi_1,$$

where both $\widehat{n}(\theta)$ and π_1 are the same as in the case without guarantees. Following the same steps as in the proof of Proposition 1, we obtain the expression (26) in the proposition.

To complete the proof, we need to compute

$$\frac{\partial \theta_\delta^*}{\partial q} = \frac{\underline{\theta} \left(r_2 - \delta \right)}{\left[\left(q r_2 - \pi_1 \frac{1-k}{L} \right) + \delta \left(\frac{1-k}{L} - q \right) \right]^2} \left[1 - \frac{\left(q r_2 - \pi_1 \right) + \delta \left(1 - q \right)}{\left(q r_2 - \pi_1 \frac{1-k}{L} \right) + \delta \left(\frac{1-k}{L} - q \right)} \right] \tag{54}$$

$$= - \frac{\left(r_2 - \delta \right) \underline{\theta}}{\left(q r_2 - \pi_1 \frac{1-k}{L} \right) + \delta \left(\frac{1-k}{L} - q \right)} \left(\pi_1 - \delta \right) \left(\frac{1-k}{L} - 1 \right) < 0,$$

and

$$\frac{\partial \theta_\delta^*}{\partial \delta} = \frac{\underline{\theta}}{\left(q r_2 - \pi_1 \frac{1-k}{L} \right) + \delta \left(\frac{1-k}{L} - q \right)} \left[(1-q) - \frac{\left(q r_2 - \pi_1 \right) + \delta \left(1 - q \right)}{\left(q r_2 - \pi_1 \frac{1-k}{L} \right) + \delta \left(\frac{1-k}{L} - q \right)} \left(\frac{1-k}{L} - q \right) \right] \tag{55}$$

$$= - \frac{q \underline{\theta} \left(r_2 - \pi_1 \right) \left(\frac{1-k}{L} - 1 \right)}{\left(q r_2 - \pi_1 \frac{1-k}{L} \right) + \delta \left(\frac{1-k}{L} - q \right)} < 0,$$

since $\pi_1 > L > \delta$ and $1-k > L$ and the proposition follows. $\square$

Proof of Proposition 13: The bank's optimal choice of q solves

$$\alpha \int_0^{\underline{\theta}_\delta} R\theta \left(1 - \frac{1-k}{L}\right) d\theta + \alpha \int_{\underline{\theta}_\delta}^1 \left[R\theta - (1-k)\, r_2\right] d\theta + (1-\alpha) \int_1^2 \left[R - (1-k)\, r_2\right] d\theta - cq = 0,$$

when $1-k \le L$ and

$$\alpha \int_{\theta_\delta^*}^1 \left[R\theta - (1-k)\, r_2\right] d\theta + (1-\alpha) \int_1^2 \left[R - (1-k)\, r_2\right] d\theta - \alpha \frac{\partial \theta_\delta^*}{\partial q} q \left[R\theta_\delta^* - (1-k)\, r_2\right] - cq = 0, \quad (56)$$

when $1-k > L$, which are obtained differentiating (27) with respect to q. When $1-k \le L$, the run threshold $\underline{\theta}_\delta$ is not affected by the deposit insurance δ as shown in the proof of Proposition 12. Hence, $\underline{q}_\delta$ is not affected by δ.

Consider now the case where $1-k > L$. In this case, the run threshold is θ_δ^* as characterized in (26). We use the implicit function theorem to compute $\frac{dq_\delta^*}{d\delta}$. Denote the expression in (56) as $FOC_{q_\delta^*} = 0$. It follows that:

$$\frac{dq_\delta^*}{d\delta} = -\frac{\frac{\partial FOC_{q_\delta^*}}{\partial \delta}}{\frac{\partial FOC_{q_\delta^*}}{\partial q}}.$$

The denominator $\frac{\partial FOC_{q_\delta^*}}{\partial q} < 0$ as q_δ^* is an interior solution. Hence, the sign of $\frac{dq_\delta^*}{d\delta}$ is equal to the sign of

$$\frac{\partial FOC_{q_\delta^*}}{\partial \delta} = -\alpha \frac{\partial \theta_\delta^*}{\partial \delta} \left[R\theta_\delta^* - (1-k)\, r_2\right] - \alpha \frac{\partial^2 \theta_\delta^*}{\partial q \partial \delta} q \left[R\theta_\delta^* - (1-k)\, r_2\right] - \alpha \frac{\partial \theta_\delta^*}{\partial q} q \frac{\partial \theta_\delta^*}{\partial \delta} R$$

All terms in the expression for $\frac{\partial FOC_{q_\delta^*}}{\partial \delta}$ are negative except the first one. We show next that the first term is dominated by the second, so that overall $\frac{\partial FOC_{q_\delta^*}}{\partial \delta} < 0$. To do so, we need to show that $q \left|\frac{\partial^2 \theta_\delta^*}{\partial q \partial \delta}\right| > \left|\frac{\partial \theta_\delta^*}{\partial \delta}\right|$. Recall that $\frac{\partial \theta_\delta^*}{\partial q}$ is given in (54). Differentiating $\frac{\partial \theta_\delta^*}{\partial q}$ with respect to δ, we obtain:

$$\begin{aligned}
\frac{\partial^2 \theta_\delta^*}{\partial q \partial \delta} = {} & \frac{\partial A}{\partial \delta} \underline{\theta} \left[1 - \frac{(qr_2 - \pi_1) + \delta (1-q)}{\left(qr_2 - \pi_1 \frac{1-k}{L}\right) + \delta \left(\frac{1-k}{L} - q\right)}\right] \\
& - A\underline{\theta} \left[(1-q) - \frac{(qr_2 - \pi_1) + \delta (1-q)}{\left(qr_2 - \pi_1 \frac{1-k}{L}\right) + \delta \left(\frac{1-k}{L} - q\right)} \left(\frac{1-k}{L} - q\right)\right],
\end{aligned}$$

where $A \equiv \frac{r_2 - \delta}{\left(qr_2 - \pi_1 \frac{1-k}{L}\right) + \delta\left(\frac{1-k}{L} - q\right)}$ and so

$$\frac{\partial A}{\partial \delta} = \frac{-\left(qr_2 - \pi_1 \frac{1-k}{L}\right) - \delta \left(\frac{1-k}{L} - q\right) - r_2 \left(\frac{1-k}{L} - q\right) + \delta \left(\frac{1-k}{L} - q\right)}{\left[\left(qr_2 - \pi_1 \frac{1-k}{L}\right) + \delta \left(\frac{1-k}{L} - q\right)\right]^2} = \frac{-\left(qr_2 - \pi_1 \frac{1-k}{L}\right) - r_2 \left(\frac{1-k}{L} - q\right)}{\left[\left(qr_2 - \pi_1 \frac{1-k}{L}\right) + \delta \left(\frac{1-k}{L} - q\right)\right]^2} < 0.$$

Using (55) and (54), the expression above can be rearranged as follows:

$$\frac{\partial^2\theta_\delta^*}{\partial q\partial\delta} = \frac{\partial A}{\partial\delta}\frac{\partial\theta_\delta^*}{\partial\delta}\frac{1}{A} - A\frac{\partial\theta_\delta^*}{\partial\delta} > 0,$$

since $\frac{\partial\theta_\delta^*}{\partial\delta} < 0$, $\frac{\partial A}{\partial\delta} < 0$ and $\frac{\partial\theta_\delta^*}{\partial\delta} < 0$. It is easy to see that $qA > 1$ since $\pi_1 > \delta$. Hence, $q\frac{\partial^2\theta_\delta^*}{\partial\delta\partial q} > \left|\frac{\partial\theta_\delta^*}{\partial\delta}\right|$ and so $\frac{dq_\delta^*}{d\delta} < 0$ as desired. □

Proof of Proposition 14: The proof follows the same steps as the proof of Proposition 12. Consider first the case when $1-k \leq L$. Since $\delta < 1$ and $\frac{Rx}{1-k} < 1$, the threshold $\underline{\theta}_{\delta x}$ under which withdrawing at date 1 is a dominant strategy is still given by $\underline{\theta}_x^P = \underline{\theta} - x$, as characterized in Section 6.2.

Consider now the case when $1-k > L$. Following the same steps as in the proof of Proposition 12, the threshold $\theta_{\delta x}^*$ corresponds to the solution to

$$\int_0^{\widehat{n}_x(\theta)} qr_2 dn + \int_{\widehat{n}(\theta)}^1 q\delta dn + \int_0^1 (1-q)\,\delta dn = \pi_1.$$

This is the same as the expression in the proof of Proposition 12, with the only difference that we have now $\widehat{n}_x(\theta) = \frac{R(\theta+x)-(1-k)r_2}{R(\theta+x)\frac{(1-k)}{L}-(1-k)r_2}$ instead of $\widehat{n}(\theta)$. Substituting in the expression above $\widehat{n}_x(\theta)$ and solving with respect to θ, we obtain the expression (28) in the proposition. It is easy to see that the comparative statics with respect to q and δ is the same as in the proof of Proposition 12. Furthermore, it is straightforward that $\frac{\partial\theta_{x\delta}^*}{\partial x} = -1$, which completes the proof. □

Proof of Proposition 15: We consider separately the case when $1-k \leq L$ and when $1-k > L$. We start from the former. From Proposition 14, the run threshold when $1-k \leq L$ is the same as in the economy without deposit insurance. Since deposit insurance only affects bank profits via the run threshold, the FOC for q is the same as (16), thus implying that $\frac{dq_{\delta x}}{dx}$ is the same as the one characterized in Proposition 6.

Consider now the case when $1-k > L$. Again, as the deposit insurance only affects bank profits via the run threshold, $q_{\delta x}^*$ is given by the solution to (17), but with $\theta_{x\delta}^*$ instead of θ_x^*. As in the proof of Proposition 6, the sign of $\frac{dq_{\delta x}^*}{dx}$ is given by the sign of the derivative of the FOC for q with respect to x, which is equal to:

$$-\alpha\left[\frac{\partial\theta_{\delta x}^*}{\partial x} + \frac{\partial^2\theta_{\delta x}^*}{\partial q\partial x}q\right]\left[R\left(\theta_{\delta x}^* + x\right) - (1-k)\,r_2\right] + \alpha\int_{\theta_{\delta x}^*}^{1-x} Rd\theta - \alpha\frac{\partial\theta_{\delta x}^*}{\partial q}qR\left[\frac{\partial\theta_{\delta x}^*}{\partial x} + 1\right].$$

From Proposition 14, we know that $\frac{\partial\theta_{\delta x}^*}{\partial x} = -1$, which, in turn, implies that $\frac{\partial^2\theta_{\delta x}^*}{\partial q\partial x} = \frac{\partial^2\theta_{\delta x}^*}{\partial x\partial q} = 0$. Then, the expression above simplifies to

$$-\alpha\frac{\partial\theta_{\delta x}^*}{\partial x}\left[R\left(\theta_{\delta x}^* + x\right) - (1-k)\,r_2\right] + \alpha\int_{\theta_{\delta x}^*}^{1-x} Rd\theta > 0.$$

Hence, $\frac{dq^*_{\delta x}}{dx} > 0$ and the proposition follows. $\square$

12 Appendix B: Alternative bankruptcy cost assumptions

In this section, we modify our assumption concerning the application of bankruptcy costs. Specifically, we consider two polar cases. First, we consider an economy in which full bankruptcy costs are also present at date 1. Second, we replicate the analysis in the absence of bankruptcy costs, which implies that depositors receive a pro-rata share of the bank available resources both at date 1 and 2.

12.1 Bankruptcy costs at date 1 and 2

In this section, we are going to introduce bankruptcy costs at date 1 so that, whenever the bank is not able to repay the promised repayment $r_1 = 1$ to all withdrawing depositors, they get a zero repayment. This modification alters the expected payoff at date 1, which we denoted as π_1 in the main text as follows:

$$\pi_1^{B_1} = \int_0^{\overline{n}=\frac{L}{(1-k)r_1}} r_1 dn + \int_{\overline{n}}^1 0 dn = \frac{L}{1-k}.$$

We consider separately the case in which the guarantees is lost in the bankruptcy procedure and that in which it is instead bankruptcy protected. We start from the former.

12.1.1 First-loss guarantee scheme

In this section, we consider the scenario in which the guarantees is lost in the bankruptcy procedure. The derivations of the run thresholds $\underline{\theta}_x$ and θ^*_x are as in the main text, with the only difference that in the expression for θ^*_x, we have $\pi_1^{B_1}$ instead of π_1. The same applies to the choice of the underwriting effort, which is still given by $\underline{q}_x$ as a solution to

$$\frac{1}{2}\int_0^{\underline{\theta}_x} R(\theta+x)\left(1-\frac{(1-k)}{L}\right)d\theta+\frac{1}{2}\int_{\underline{\theta}_x}^{1-x}\left[R(\theta+x)-(1-k)r_2\right]d\theta+\frac{1}{2}\int_{1-x}^{2}\left[R-(1-k)r_2\right]d\theta-cq=0 \tag{57}$$

when $1-k \leq L$ and q^*_x when $1-k > L$ as a solution to

$$\frac{1}{2}\int_{\theta^*_x}^{1-x}\left[R(\theta+x)-(1-k)r_2\right]d\theta+\frac{1}{2}\int_{1-x}^{2}\left[R-(1-k)r_2\right]d\theta-\frac{1}{2}\frac{\partial\theta^*_x}{\partial q}q\left[R(\theta^*_x+x)-(1-k)r_2\right]-cq=0. \tag{58}$$

Since $\underline{\theta}_x$ is not affected by the introduction of bankruptcy costs at date 1, the results concerning the effect of the introduction of the guarantees on $\underline{q}_x$ go through as in the main text whenever $(1-k)r_1 \leq L$.

Consider now the case in which $1-k > L$, for which the relevant run threshold θ_x^* is now slightly different from the one in the main text since it includes $\pi_1^{B_1}$ instead of π_1. Differentiating (58) with respect to x, we obtain

$$-\frac{1}{2}\frac{\partial\theta_x^*}{\partial x}q\left[R\left(\theta_x^*+x\right)-(1-k)\,r_2\right]+\frac{1}{2}\int_{\theta_x^*}^{1-x}Rd\theta-\frac{1}{2}\frac{\partial^2\theta_x^*}{\partial q\partial x}q\left[R\left(\theta_x^*+x\right)-(1-k)\,r_2\right]$$
$$-\frac{1}{2}\frac{\partial\theta_x^*}{\partial q}q\frac{\partial\theta_x^*}{\partial x}R-\frac{1}{2}\frac{\partial\theta_x^*}{\partial q}qR.$$

Yes, it is easy to see that the change in depositors' expected repayment at date 1 does not affect the properties of the run threshold θ_x^* since $\frac{\partial\theta_x^*}{\partial x}=-1$ and $\frac{\partial^2\theta_x^*}{\partial q\partial x}=\frac{\partial^2\theta_x^*}{\partial x\partial q}=0$. Hence, the expression above simplifies to

$$+\frac{1}{2}q\left[R\left(\theta_x^*+x\right)-(1-k)\,r_2\right]+\frac{1}{2}\int_{\theta_x^*}^{1-x}Rd\theta>0,$$

which is the same we have in the main text. Hence, the result that $\frac{\partial q_x^*}{\partial x}>0$ holds true also in the case with bankruptcy costs at date 1.

12.1.2 Bankruptcy-protected guarantee scheme

In this case, we consider the scenario in which the guarantees x is protected from bankruptcy. As no guarantee is paid at date 1, the introduction of the bankruptcy costs in case the bank is unable to repay $r_1=1$ only alters the expected payoff at date 1 and does not directly interact with the guarantee. This implies that the run risk in the presence of a first-loss guarantee x whose transfers are protected in bankruptcy is still given by $\underline{\theta}_x^P=\underline{\theta}_x$ when $1-k\leq L$, which, in turn, implies that the result still goes through as in the main text for well-capitalized banks.

Consider now the case when $1-k>L$. The run threshold θ_x^{*P} is given by solution to

$$\pi_1^{B_1}=\int_0^{\widehat{n}_x(\theta)}qr_2dn+\int_{\widehat{n}_x(\theta)}^{\overline{n}}q\frac{Rx\left(1-n\frac{(1-k)}{L}\right)}{(1-n)\,(1-k)}dn+\int_0^{\overline{n}}(1-q)\,\frac{Rx\left(1-n\frac{(1-k)}{L}\right)}{(1-n)\,(1-k)}dn. \tag{59}$$

It is easy to see that, since the introduction of bankruptcy costs at date 1 only affects depositors' expected payoff at date , as given by the LHS in the expression above, the properties of the run threshold are the same as in the main text, i.e., $\frac{\partial\theta_x^{*P}}{\partial x}$, $\frac{\partial\theta_x^{*P}}{\partial q}$ and $\frac{\partial^2\theta_x^{*P}}{\partial x\partial q}$ are as in the baseline model. Using the same arguments as in the previous section, it follows that the results of our baseline model go through also when introducing bankruptcy costs at date 1.

12.2 No bankruptcy costs at either date

In this section, we assume that, contrary to the main text, there are no bankruptcy costs at either date 1 or at date 2. This implies that depositors receive a pro-rata share of the bank's available resources when the bank is unable to repay the promised payment both at dates 1 and 2. As we show below, while this modification complicates the analysis substantially, all the main results go through. In what follows, we proceed in steps. First, we characterize the thresholds when the bank either experiences a run or goes bankrupt at date 2. Then, we characterize the effect of a loan guarantee on bank underwriting incentives.

In the absence of bankruptcy costs at date 2, bank profits depend on whether the bank faces a run (either fundamental or panic driven) or whether it goes bankrupt at date 2. The following lemma describes the thresholds characterizing these three different cases.

Lemma 4 *Denote θ_x^B as the threshold below which the bank is unable to repay the promised date 2 payment to depositors, and $\underline{\theta}_x$ and θ_x^* as the run thresholds when $1-k \leq L$ and $1-k > L$, respectively. Then, we have the following:*

1. *The solvency threshold θ_x^B solves $R(\theta + x) - (1-k)r_2 = 0$ and is equal to*

$$\theta_x^B = \frac{(1-k)\,r_2}{R} - x. \tag{60}$$

2. *The fundamental run threshold $\underline{\theta}_x$ solves $q\frac{R(\theta+x)}{1-k} = 1$ and is given by*

$$\underline{\theta}_x = \frac{(1-k)}{qR} - x. \tag{61}$$

3. The panic run threshold θ_x^* solves

$$\int_0^{\widehat{n}_x(\theta)} qr_2 dn + \int_{\widehat{n}_x(\theta)}^{\overline{n}} q\frac{R(\theta+x)\left(1 - n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn + (1-q)\int_0^{\overline{n}} \frac{Rx\left(1 - n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn - \pi_1 = 0, \tag{62}$$

where $\widehat{n}_x(\theta^*)$ and $\overline{n}$ are defined in the proof and $\pi_1 = \int_0^{\overline{n}} dn + \int_{\overline{n}}^1 \frac{L}{(1-k)n}dn$.

Proof. 4: The proof proceeds in steps.

Step 1: Characterization of the solvency threshold θ_x^B and the fundamental run threshold $\underline{\theta}_x$: The characterization of the solvency threshold θ_x^B is straightforward and follows directly from the condition $R(\theta + x) - (1-k)r_2 = 0$. As in the baseline model, we pin down the fundamental run threshold as the upper bound of the lower dominance region, i.e., the region where running is a dominant strategy. Under the assumption that no one else runs, a depositor never finds it optimal to run when $\theta > \theta_x^B$ as in this case he expects to receive $qr_2 \geq 1$. This implies that running can only be a dominant strategy for lower values of θ, i.e., $\theta < \underline{\theta}_x < \theta_x^B$. When $\theta < \theta_x^B$, a depositor expects to receive the pro-rata share $q\frac{R(\theta+x)}{1-k}$ at date 2. Running is then optimal when the pro-rata share falls below the date 1 repayment. At the threshold $\underline{\theta}_x$, the date 1 and 2 repayments are identical as given by the condition $q\frac{R(\theta+x)}{1-k} = 1$.

Step 2: Characterization of the panic run threshold θ_x^*: We follow the same steps as in the main text, starting with the characterization of the two conditions pinning down $\{s_x^*, \theta_x^*\}$:

$$R\left(\theta_x^* + x\right)\left(1 - \frac{n\left(\theta_x^*, s_x^*\right)(1-k)}{L}\right) - \left(1 - n\left(\theta_x^*, s_x^*\right)\right)(1-k)\, r_2 = 0 \tag{63}$$

and

$$qr_2 \Pr\left(\theta > \theta_x^* \middle| s_x^*\right) + qRE[\theta_x^* < \theta > \theta_n | \, s_x^*] \frac{\left(1 - n(\theta, s_x^*)\frac{(1-k)}{L}\right)}{(1 - n(\theta, s_x^*))(1-k)}$$

$$+ (1-q)\frac{Rx\left(1 - n(\theta, s_x^*)\frac{(1-k)}{L}\right)}{(1-n(\theta, s_x^*))(1-k)} \Pr\left(\theta < \theta_n \middle| s_x^*\right) = 1 \Pr\left(\theta > \theta_n \middle| s_x^*\right) + \frac{L}{(1-k)n(\theta, s_x^*)} \Pr\left(\theta < \theta_n \middle| s_x^*\right), \tag{64}$$

where $\theta_n = s_x^* + \varepsilon - 2\varepsilon\frac{L}{1-k}$ represents the level of θ for which the bank liquidates the entire portfolio at date 1 and, thus, is equal to the solution to

$$n\left(\theta_n, s_x^*\right)(1-k) = L.$$

Condition (63) identifies the level of fundamentals, θ_x^*, at which the bank is at the brink of insolvency at date 2 when $n\left(\theta_x^*, s_x^*\right) > 0$ depositors run, for given s_x^*. Condition (64) is depositors' indifference condition: the LHS represents a depositor's expected utility from withdrawing at date 2, while the RHS represents the expected utility from withdrawing at date 1. This condition pins down s_x^* given $\theta_x^*\left(s_x^*\right)$ from (63), so that together the two equations characterize the equilibrium withdrawal decisions $\{s_x^*, \theta_x^*\}$. To obtain the expression (62) in the proposition, where $\widehat{n}_x\left(\theta_x^*\right)$ solves (63) and $\overline{n}$ solves

$$(1-k)n = L,$$

we perform a change of variable by defining $\theta_x^*(n) = s_x^* + \varepsilon(1-2n)$ and then take the limit when $\varepsilon \to 0$, so that $\theta_x^*(n) \to s_x^*$. ■

Having characterized the two run thresholds $\underline{\theta}_x$ and θ_x^*, we can follows the same steps as in the baseline model to establish that the relevant threshold is $\underline{\theta}_x$ when $1-k \leq L$ and θ_x^* when $1-k > L$. To this end, we start assuming $1-k = L$. In this case $\pi_1 \equiv \int_0^{\overline{n}} dn + \int_{\overline{n}}^1 \frac{L}{(1-k)n} dn = 1$ and (63) simplifies to

$$(1-n)\left[R\theta - (1-k)\, r_2\right],$$

which is positive for $\theta > \theta_x^B$ and negative for $\theta < \theta_x^B$ for any $n < 1$. It follows that running is optimal when $\theta < \underline{\theta}_x$, irrespective of n. This implies that the relevant run threshold is $\underline{\theta}_x$ when

$1-k=L$. Since $\underline{\theta}_x$ is decreasing in $1-k$, condition (63) becomes less binding for any n when $1-k$ falls below L. This implies that $\underline{\theta}_x$ is still the relevant run threshold when $1-k<L$.

Consider now the case when $1-k>L$. Since (63) is increasing in θ, it follows that $\theta_x^*>\underline{\theta}_x$ when $1-k>L$. However, as $1-k\to L$, we know that $qr_2\to 1$ from depositors' participation constraint being satisfied with equality. This completes the proof and the lemma follows. $\square$

The lemma shows that, unlike in the baseline model, a depositor's decision to run is no longer driven by the bank's solvency condition when $1-k\le L$. This occurs because, when the bank is unable to repay the promised r_2 at date 2, depositors receive a pro-rata share of the bank's available resources. This implies that, relative to the baseline model, they have less incentives to run and, as a result, $\underline{\theta}_x<\theta_x^B$.

When $1-k>L$, depositors find it optimal to run when θ falls below θ_x^*. Again, since depositors receive a pro-rata share if the bank goes bankrupt at date 2, the panic run threshold is now lower than in the baseline model. It is also potentially lower than the solvency threshold θ^B. Whether $\theta_x^B\gtrless\theta_x^*$ depends on the level of bank capital. In the limiting case when $k\to 0$ and $x\to 0$, $\theta_x^*\to 1$, while $\theta_x^B<<1$. Hence, $\theta_x^*>\theta_x^B$ in that case. At the other extreme, when $1-k\to L$, both $\theta_x^*\to\underline{\theta}_x$ and $\theta_x^B\to\underline{\theta}_x$ since, in this case, $qr_2\to 1$. This means that $\theta_x^*\to\theta_x^B$. For intermediate values of k, there exist parameters consistent with $\theta_x^*>\theta_x^B$ and with $\theta_x^*<\theta_x^B$. We thus proceed to analyze the effect of the loan guarantee on bank underwriting standards in either case.

As in the baseline model, the bank chooses q so as to maximize expected profits. When $1-k\le L$, the bank sets q_x as the solution to

$$\max_q \Pi=\alpha\int_0^{\underline{\theta}_x} qR(\theta+x)\left(1-\frac{1-k}{L}\right)d\theta+\alpha\int_{\underline{\theta}_x}^{1-x} q\left[R(\theta+x)-(1-k)\,r_2\right]d\theta$$
$$+\alpha\int_{1-x}^{1} q\left[R-(1-k)\,r_2\right]d\theta+(1-\alpha)\int_1^2 q\left[R-(1-k)\,r_2\right]d\theta-\frac{cq^2}{2}. \tag{65}$$

When $1-k>L$, the bank chooses q_x as the solution to

$$\max_q \Pi=\alpha\int_{\max\{\theta_x^B,\theta_x^*\}}^{1-x} q\left[R(\theta+x)-(1-k)\,r_2\right]d\theta+\alpha\int_{1-x}^{1} q\left[R-(1-k)\,r_2\right]d\theta$$
$$+(1-\alpha)\int_1^2 q\left[R-(1-k)\,r_2\right]d\theta-\frac{cq^2}{2} \tag{66}$$

We have the following result.

Proposition 16 *The impact of a first-loss guarantee x on bank underwriting effort when there are no bankruptcy costs is as follows:*

a) When $1-k\le L$, introducing x increases bank underwriting effort, i.e., $\frac{dq_x}{dx}>0$;

b) When $1-k>L$, the bank fails if $\theta<\max\left\{\theta_x^B,\theta_x^*\right\}$. There exists a value of k denoted as $\widehat{k}_x<1-L$ such that introducing x reduces bank effort for $k<\widehat{k}_x$, while increasing it as $k\to 1-L$: $\frac{dq_x}{dx}<0$ for $k<\widehat{k}_x$ and $\frac{dq_x}{dx}>0$ for $k\to 1-L$.

Proof. 16: We consider separately the three cases, starting from the case when $1-k\leq L$. Differentiating (65) with respect to q, we obtain the FOC_{q_x} as follows:

$$\begin{gathered}\alpha\int_0^{\underline{\theta}_x}R\left(\theta+x\right)\left(1-\tfrac{1-k}{L}\right)d\theta+\alpha\int_{\underline{\theta}_x}^{1-x}\left[R\left(\theta+x\right)-\left(1-k\right)r_2\right]d\theta+\alpha\int_{1-x}^{1}\left[R-\left(1-k\right)r_2\right]d\theta\\+\left(1-\alpha\right)\int_1^2\left[R-\left(1-k\right)r_2\right]d\theta+\alpha\tfrac{\partial\underline{\theta}_x}{\partial q}qR\left(\underline{\theta}_x+x\right)\left(1-\tfrac{1-k}{L}\right)-cq=0\end{gathered},$$

where $\frac{\partial\underline{\theta}_x}{\partial q}=-\frac{(1-k)}{q^2R}$. The effect of x on q_x can be computed using the implicit function theorem, i.e., $\frac{dq_x}{dx}=-\frac{\frac{\partial FOC_{q_x}}{\partial x}}{\frac{\partial FOC_{q_x}}{\partial q}}$. Given that q_x is an interior solution and so $\frac{\partial FOC_{q_x}}{\partial q}<0$, the sign of $\frac{dq_x}{dx}$ is equal to the sign of $\frac{\partial FOC_{q_x}}{\partial x}$. This is equal to the following expression:

$$\begin{aligned}\frac{\partial FOC_{q_x}}{\partial x}=&\alpha\int_0^{\underline{\theta}_x}R\left(1-\frac{1-k}{L}\right)d\theta+\alpha\int_{\underline{\theta}_x}^{1-x}\left[R-\left(1-k\right)r_2\right]d\theta+\alpha\frac{\partial\underline{\theta}_x}{\partial x}R\left(\underline{\theta}_x+x\right)\left(1-\frac{1-k}{L}\right)\\&+\alpha\frac{\partial\underline{\theta}_x}{\partial q}qR\frac{\partial\underline{\theta}_x}{\partial x}\left(1-\frac{1-k}{L}\right)+\alpha\frac{\partial\underline{\theta}_x}{\partial q}qR\left(1-\frac{1-k}{L}\right)\\&=\alpha\int_0^{\underline{\theta}_x}R\left(1-\frac{1-k}{L}\right)d\theta+\alpha\int_{\underline{\theta}_x}^{1-x}\left[R-\left(1-k\right)r_2\right]d\theta-\alpha R\left(\underline{\theta}_x+x\right)\left(1-\frac{1-k}{L}\right)\\&=\alpha\int_{\underline{\theta}_x}^{1-x}\left(R-\left(1-k\right)r_2\right)d\theta-\alpha Rx\left(1-\frac{1-k}{L}\right).\end{aligned}$$

Using $\underline{\theta}_x=\frac{(1-k)}{qR}-x$, the expression above can be further rearranged as follows:

$$\begin{aligned}\frac{\partial FOC_{q_x}}{\partial x}&=\alpha R\left[1-x-\frac{(1-k)}{qR}+x-x+\frac{(1-k)}{L}\right]-\alpha\left(1-k\right)r_2\left[1-x-\frac{(1-k)}{qR}+x\right]\\&=\alpha R\left[1-x-\frac{(1-k)}{qR}+\frac{(1-k)}{L}\right]-\alpha\left(1-k\right)r_2\left[1-\frac{(1-k)}{qR}\right],\end{aligned}$$

where $1-x-\frac{(1-k)}{qR}+\frac{(1-k)}{L}>1-\frac{(1-k)}{qR}>0$ for all $x<\frac{(1-k)}{L}$. Since we restrict our analysis to the case where $x<\frac{1-k}{R}$, and since $\frac{1-k}{R}<\frac{1-k}{L}$, it follows that $\frac{\partial FOC_{q_x}}{\partial x}>0$ and so $\frac{dq_x}{dx}>0$ when $1-k\leq L$. ■

Consider now the case when $1-k>L$. We start by assuming that $\theta_x^*>\theta_x^B$. In this case, the FOC_q is equal to

$$\begin{gathered}\alpha\int_{\theta_x^*}^{1-x}\left[R\left(\theta+x\right)-\left(1-k\right)r_2\right]d\theta+\alpha\int_{1-x}^{1}\left[R-\left(1-k\right)r_2\right]d\theta+\left(1-\alpha\right)\int_1^2\left[R-\left(1-k\right)r_2\right]d\theta\\-\alpha\tfrac{\partial\theta_x^*}{\partial q}q\left[R\left(\theta_x^*+x\right)-\left(1-k\right)r_2\right]-cq=0.\end{gathered}$$

Then, differentiating the expression above with respect to x, we obtain

$$-\alpha\frac{\partial\theta_x^*}{\partial x}\left[R\left(\theta_x^*+x\right)-\left(1-k\right)r_2\right]+\alpha\int_{\theta_x^*}^{1-x}Rd\theta-\alpha\frac{\partial\theta_x^*}{\partial q}\frac{\partial\theta_x^*}{\partial x}qR-\alpha\frac{\partial\theta_x^*}{\partial q}qR-\alpha\frac{\partial^2\theta_x^*}{\partial q\partial x}q\left[R\left(\theta_x^*+x\right)-\left(1-k\right)r_2\right],$$

which can be further rearranged as

$$-\alpha\left[\frac{\partial\theta_x^*}{\partial x}+\frac{\partial^2\theta_x^*}{\partial q\partial x}q\right]\left[R\left(\theta_x^*+x\right)-\left(1-k\right)r_2\right]-\alpha\frac{\partial\theta_x^*}{\partial q}\frac{\partial\theta_x^*}{\partial x}qR+\alpha\int_{\theta_x^*}^{1-x}Rd\theta-\alpha\frac{\partial\theta_x^*}{\partial q}qR. \tag{67}$$

Rearranging the terms, we obtain:

$$-\alpha\left[\frac{\partial\theta_x^*}{\partial x}+\frac{\partial^2\theta_x^*}{\partial q\partial x}q\right]\left[R\left(\theta_x^*+x\right)-\left(1-k\right)r_2\right]-\alpha qR\frac{\partial\theta_x^*}{\partial q}\left(\frac{\partial\theta_x^*}{\partial x}+1\right)+\alpha\int_{\theta_x^*}^{1-x}Rd\theta \tag{68}$$

As in the case of bankruptcy-protected guarantees, we consider the limiting case when $k\rightarrow 0$ and $x\rightarrow 0$. Given the indifference condition pinning down θ_x^*, we compute $\frac{\partial\theta_x^*}{\partial x}$ as follows:

$$\frac{\partial\theta_x^*}{\partial x}=-\frac{\frac{\partial\widehat{n}_x(\theta_x^*)}{\partial x}q\left[r_2-\frac{R(\theta_x^*+x)\left(1-\widehat{n}_x(\theta_x^*)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta_x^*))(1-k)}\right]+\int_{\widehat{n}_x(\theta_x^*)}^{\overline{n}}q\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn+(1-q)\int_0^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\frac{\partial\widehat{n}_x(\theta_x^*)}{\partial\theta}q\left[r_2-\frac{R(\theta_x^*+x)\left(1-\widehat{n}_x(\theta_x^*)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta_x^*))(1-k)}\right]+\int_{\widehat{n}_x(\theta_x^*)}^{\overline{n}}q\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}$$

$$-1-\frac{1-q}{q}\frac{\int_0^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\int_{\widehat{n}_x(\theta_x^*)}^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}<-1$$

since $r_2-\frac{R(\theta_x^*+x)\left(1-\widehat{n}_x(\theta_x^*)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta_x^*))(1-k)}=0$ from the definition of $\widehat{n}_x\left(\theta_x^*\right)$ and $\frac{\partial\widehat{n}_x(\theta_x^*)}{\partial x}\equiv\frac{\partial\widehat{n}_x(\theta_x^*)}{\partial\theta}$ being equal to

$$\frac{\partial\widehat{n}_x\left(\theta_x^*\right)}{\partial x}=\frac{R}{R\left(\theta_x^*+x\right)\frac{(1-k)}{L}-\left(1-k\right)r_2}\left[1-\widehat{n}_x\left(\theta_x^*\right)\frac{\left(1-k\right)}{L}\right]>0.$$

Similarly, we can compute

$$\frac{\partial\theta_x^*}{\partial q}=-\frac{\int_0^{\widehat{n}_x(\theta)}r_2dn+\int_{\widehat{n}_x(\theta)}^{\overline{n}}\frac{R(\theta+x)\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn-\int_0^{\overline{n}}\frac{Rx\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\frac{\partial\widehat{n}_x(\theta_x^*)}{\partial\theta}q\left[r_2-\frac{R(\theta_x^*+x)\left(1-\widehat{n}_x(\theta_x^*)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta_x^*))(1-k)}\right]+\int_{\widehat{n}_x(\theta_x^*)}^{\overline{n}}q\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}<0,$$

which is similar to what is obtained in the baseline model.

Since $\frac{\partial\theta_x^*}{\partial x}<-1$ and $\frac{\partial\theta_x^*}{\partial q}<0$, the second term in (68) is negative. The third term, which is positive, goes to zero as $\theta_x^*\rightarrow 1$. This only leaves the first term to sign, which requires to compare the cross partial, $\frac{\partial^2\theta_x^*}{\partial q\partial x}q$ to $\frac{\partial\theta_x^*}{\partial x}$. A sufficient condition for the negative effect of the introduction of the guarantees on bank underwriting incentives, in line with the baseline model with the case of the bankruptcy-protected guarantee scheme, is that $\frac{\partial\theta_x^*}{\partial x}+\frac{\partial^2\theta_x^*}{\partial q\partial x}q>0$. To this end, we compute $\frac{\partial\theta_x^*}{\partial x\partial q}$ as follows:

$$\frac{\partial^2\theta_x^*}{\partial x\partial q}=-\frac{-q-1+q}{q^2}\frac{\int_0^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\int_{\widehat{n}_x(\theta_x^*)}^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}-\frac{(1-q)\int_0^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{q\left[\int_{\widehat{n}_x(\theta_x^*)}^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn\right]^2}\frac{\partial\widehat{n}_x(\theta_x^*)}{\partial\theta}\frac{\partial\theta_x^*}{\partial q}\frac{R\left(1-\widehat{n}_x(\theta_x^*)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta_x^*))(1-k)}$$

$$=\frac{1}{q^2}\frac{\int_0^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\int_{\widehat{n}_x(\theta_x^*)}^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}-\frac{1-q}{q}\frac{\int_0^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\left[\int_{\widehat{n}_x(\theta_x^*)}^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn\right]^2}\frac{\partial\widehat{n}_x(\theta_x^*)}{\partial\theta}\frac{\partial\theta_x^*}{\partial q}\frac{R\left(1-\widehat{n}_x(\theta_x^*)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta_x^*))(1-k)}>0.$$

Hence, to prove that $\frac{dq_x^*}{dx}<0$ when $k\to 0$ and $x\to 0$, we only need to establish that $\left|\frac{\partial^2\theta_x^*}{\partial x\partial q}\right|>\left|\frac{\partial\theta_x^*}{\partial x}\right|$. From the expression for $\frac{\partial\theta_x^*}{\partial x}$, we can write $-\frac{1-q}{q}\frac{\int_0^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\int_{\widehat{n}_x(\theta_x^*)}^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}=+1+\frac{\partial\theta_{xNB}^*}{\partial x}$ and $\frac{1}{q}\frac{\int_0^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\int_{\widehat{n}_x(\theta_x^*)}^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}=-\frac{1}{1-q}\frac{\partial\theta_x^*}{\partial x}-\frac{1}{1-q}$, which allows us to rewrite the expression above as

$$\frac{\partial^2\theta_x^*}{\partial x\partial q}=-\frac{1}{q}\frac{1}{1-q}\frac{\partial\theta_x^*}{\partial x}-\frac{1}{q(1-q)}+\frac{\partial\widehat{n}_x(\theta_x^*)}{\partial\theta}\frac{\partial\theta_x^*}{\partial q}\frac{R\left(1-\widehat{n}_x(\theta_x^*)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta_x^*))(1-k)}$$
$$+\frac{\partial\theta_x^*}{\partial x}\frac{\partial\widehat{n}_x(\theta_x^*)}{\partial\theta}\frac{\partial\theta_x^*}{\partial q}\frac{R\left(1-\widehat{n}_x(\theta_x^*)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta_x^*))(1-k)}.$$

Now, we substitute the expression for $\frac{\partial^2\theta_x^*}{\partial x\partial q}$ into the $\left[\frac{\partial\theta_x^*}{\partial x}+\frac{\partial^2\theta_x^*}{\partial q\partial x}q\right]$ in (68) and obtain

$$-\frac{q}{1-q}\frac{\partial\theta_x^*}{\partial x}-\frac{1}{(1-q)}+q\frac{\partial\widehat{n}_x(\theta_x^*)}{\partial\theta}\frac{\partial\theta_x^*}{\partial q}\frac{R\left(1-\widehat{n}_x(\theta_x^*)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta_x^*))(1-k)}\left[1+\frac{\partial\theta_x^*}{\partial x}\right].$$

The first and last terms are positive since $\frac{\partial\theta_x^*}{\partial x}<-1$. A sufficient condition for the whole expression above to be positive is that

$$\left|\frac{\partial\theta_x^*}{\partial x}\right|<\frac{1}{q}\Longleftrightarrow-\frac{\partial\theta_x^*}{\partial x}-\frac{1}{q}>0,$$

which implies that the first and second term sum up to a positive. Recall that

$$\frac{\partial\theta_x^*}{\partial x}=-1-\frac{1-q}{q}\frac{\int_0^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\int_{\widehat{n}_x(\theta_x^*)}^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}.$$

Hence we need to show that

$$1+\frac{1-q}{q}\frac{\int_0^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\int_{\widehat{n}_x(\theta_x^*)}^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}-\frac{1}{q}>0\Longleftrightarrow\frac{1-q}{q}\frac{\int_0^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\int_{\widehat{n}_x(\theta_x^*)}^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}>\frac{1-q}{q}.$$

The inequality above holds true since $\frac{\int_0^{\overline{n}} \frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn}{\int_{\widehat{n}_x(\theta_x^*)}^{\overline{n}} \frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn} > 1$. Then, it follows that $\frac{\partial \theta_x^*}{\partial x} + \frac{\partial^2 \theta_x^*}{\partial q \partial x} q > 0$ as desired and the rest of the proof for the case $k \to 0$ and $x \to 0$ follows as in the case with bankruptcy- protected guarantees.

The final case is when $\theta_x^B > \theta_x^*$, which, as argued above, can only be true for $k >> 0$. In this case, the FOC_q is equal to

$$\alpha \int_{\theta_x^B}^{1-x} \left[R(\theta + x) - (1-k) r_2\right] d\theta + \alpha \int_{1-x}^{1} \left[R - (1-k) r_2\right] + (1-\alpha) \int_1^2 \left[R - (1-k) r_2\right] d\theta - cq = 0,$$

since the derivatives of the extremes of the integrals cancel out based on the definition of θ_x^B. Taking the derivative with respect to x, we obtain

$$-\alpha \frac{\partial \theta_x^B}{\partial x} \left[R\left(\theta_x^B + x\right) - (1-k) r_2\right] d\theta + \alpha \int_{\theta_x^B}^{1-x} R d\theta > 0,$$

since $\left[R\left(\theta_x^B + x\right) - (1-k) r_2\right] = 0$ from the definition of θ_x^B. Hence, x increases underwriting standards in this case and the proposition follows. □

The results in Proposition 16 shows that the results in the main text are robust to the absence of bankruptcy costs. Specifically, the results in Proposition 16 mirror the ones in the main text for the case of bankruptcy-protected guarantees, which state that the introduction of the guarantees generally leads to an improvement in bank underwriting standards, except for those banks with a low level of capital for which the probability of a panic run is significant. Furthermore, the analysis in Proposition 16 also shows that the general beneficial effect of loan guarantees on bank underwriting incentives continue to hold also when the relevant threshold for the bank is θ_x^B. This demonstrates overall that assuming full bankruptcy costs at date 2 does not qualitatively affects the results, while it significantly simplifies the analysis.

12.3 Comparative statics of θ^* with respect to L as $k \to 0$

As $k \to 0$, the expression for θ^* in (6) becomes

$$\theta^* = \underline{\theta} \frac{q r_2 - \pi_1}{q r_2 - \pi_1 \frac{1}{L}}$$

where $\underline{\theta} = \frac{r_2}{R}$ and

$$\pi_1 = \int_0^L dn + \int_L^1 \frac{L}{n} dn = L\left(1 + \int_L^1 \frac{1}{n} dn\right).$$

Recall that θ^* is bounded above by 1 since $\theta^* < \overline{\theta} < 1$. This implies that denominator is bounded below by 0. Formally, we can then rewrite the expression for θ^* as follows:

$$\theta^* = \underline{\theta} \frac{qr_2 - \pi_1}{\max\left\{qr_2 - \pi_1 \frac{1}{L}, 0\right\}}$$

Substituting the expressions for $\underline{\theta}$ and π_1 into that for θ^* gives:

$$\theta^* = \frac{r_2}{R} \frac{qr_2 - L\left(1 + \int_L^1 \frac{1}{n} dn\right)}{\max\left\{qr_2 - L\left(1 + \int_L^1 \frac{1}{n} dn\right)\frac{1}{L}, 0\right\}} = \frac{r_2}{R} \frac{qr_2 - L\left(1 + \int_L^1 \frac{1}{n} dn\right)}{\max\left\{qr_2 - \left(1 + \int_L^1 \frac{1}{n} dn\right), 0\right\}}$$

We now take the limit of the above expression as $L \to 0$, starting with the denominator. As $L \to 0$, $1 + \int_L^1 \frac{1}{n} dn \to \infty$. Hence, the denominator goes to zero as $L \to 0$.

Consider now the numerator. It is useful to rearrange the second term as

$$L\left(1 + \int_L^1 \frac{1}{n} dn\right) = \frac{\left(1 + \int_L^1 \frac{1}{n} dn\right)}{\frac{1}{L}}.$$

Using L'Hopital's rule, the limit is equal to

$$\lim_{L \to 0} \frac{\left(1 + \int_L^1 \frac{1}{n} dn\right)}{\frac{1}{L}} = \lim_{L \to 0} \frac{-\frac{1}{L}}{-\frac{1}{L^2}} = \lim_{L \to 0} L = 0.$$

This implies that as $L \to 0$, the numerator goes to $qr_2 \geq 1$, while the denominator goes to 0. Since $\frac{r_2}{R}$ is bounded below by $\frac{1}{R} > 0$, the entire expression approaches to $+\infty$. Hence, for L small enough $\theta^* = 1$.

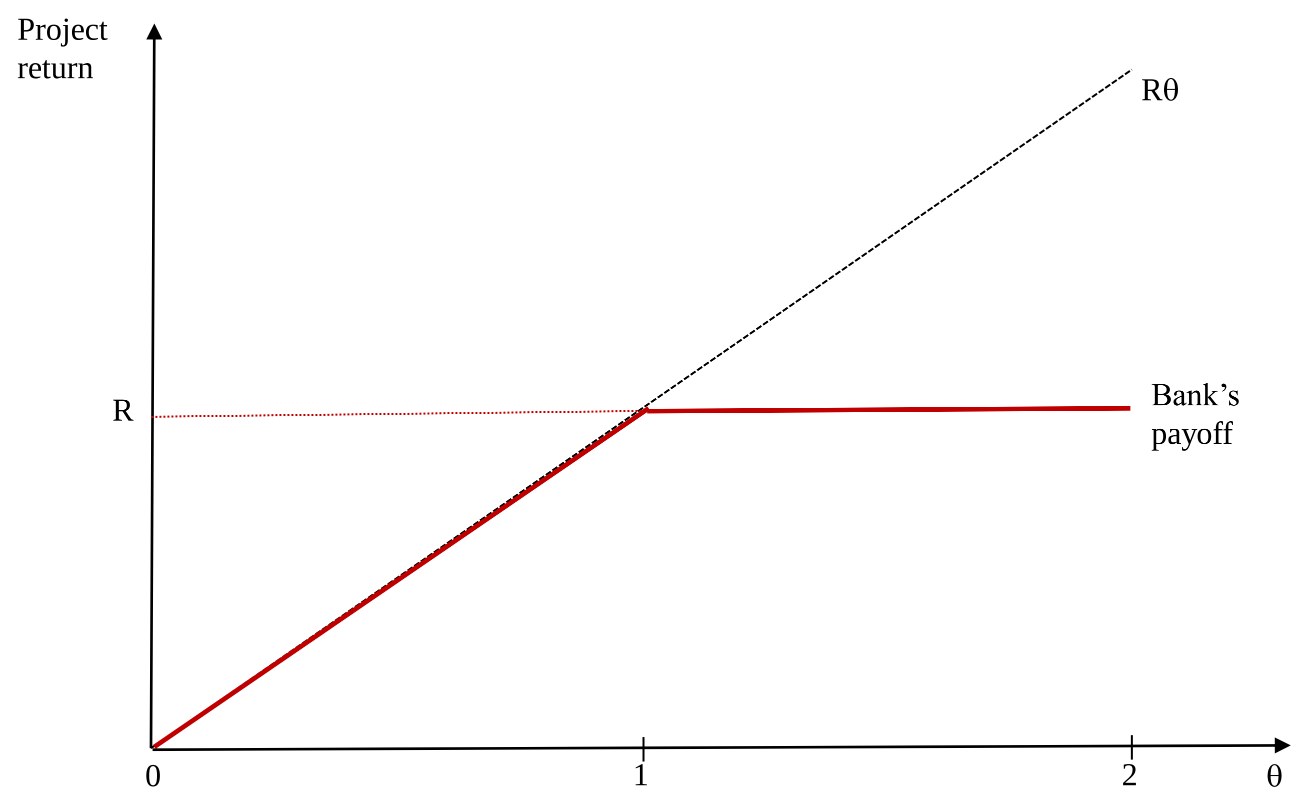


Figure 1: **Bank payoff.** The figure illustrates a bank's payoff as a function of the fundamental of the economy θ. This is equal to R for any $\theta \geq 1$ and Rθ for any $\theta < 1$.

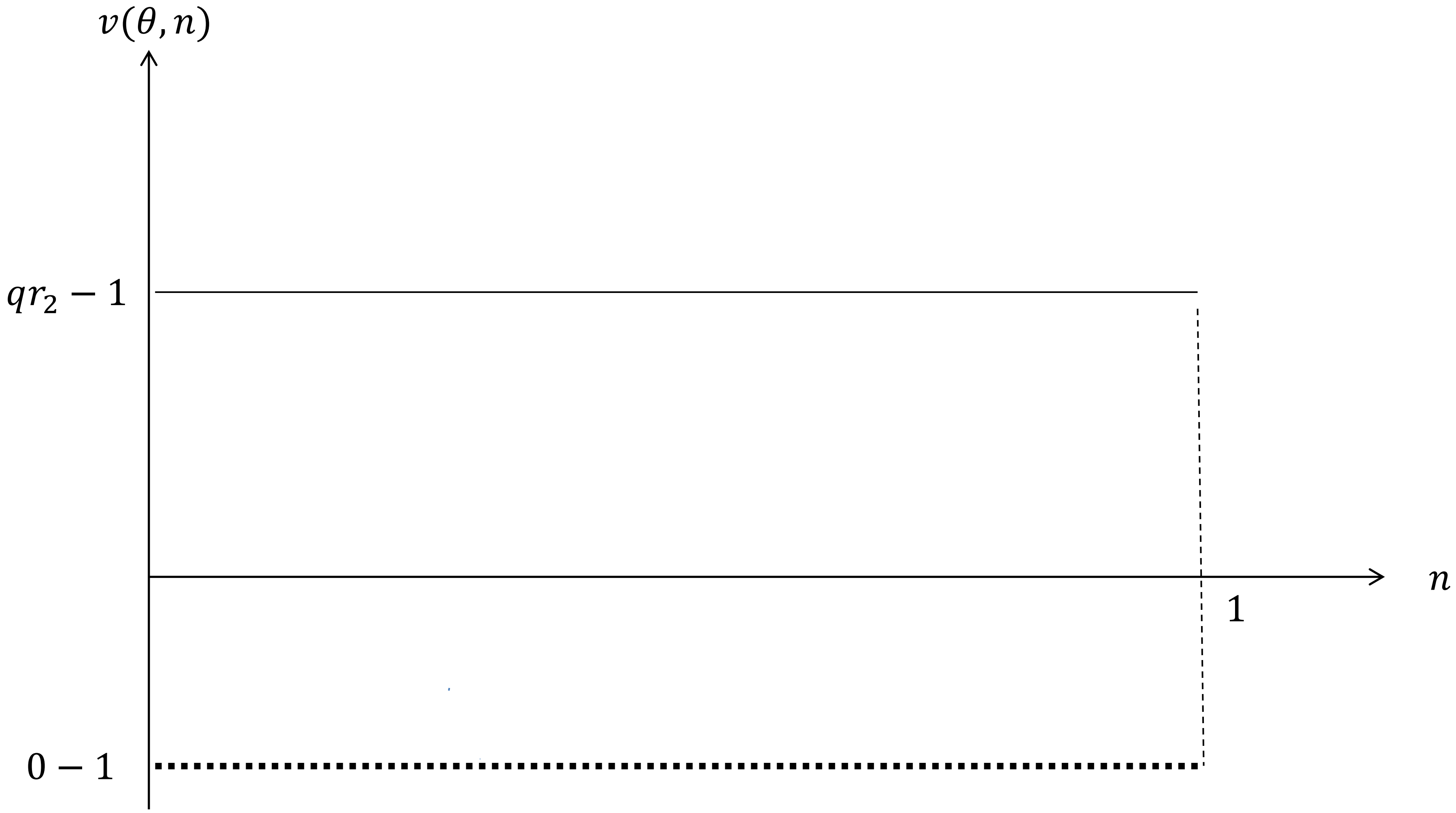


Figure 2: **Depositors' payoff differential when $(1-k) \leq L$.** The figure shows a depositor's payoff differential between withdrawing at date 2 and at date 1 as a function of the proportion of depositors withdrawing early, n, when 1-k ≤ L. The payoff differential is independent of n and only depends on whether the bank is solvent at date 2. The solid line represents the utility differential when the bank is solvent at date 2, while the dotted one captures the utility differential when the bank does not have enough resources to repay depositors even if no one runs.

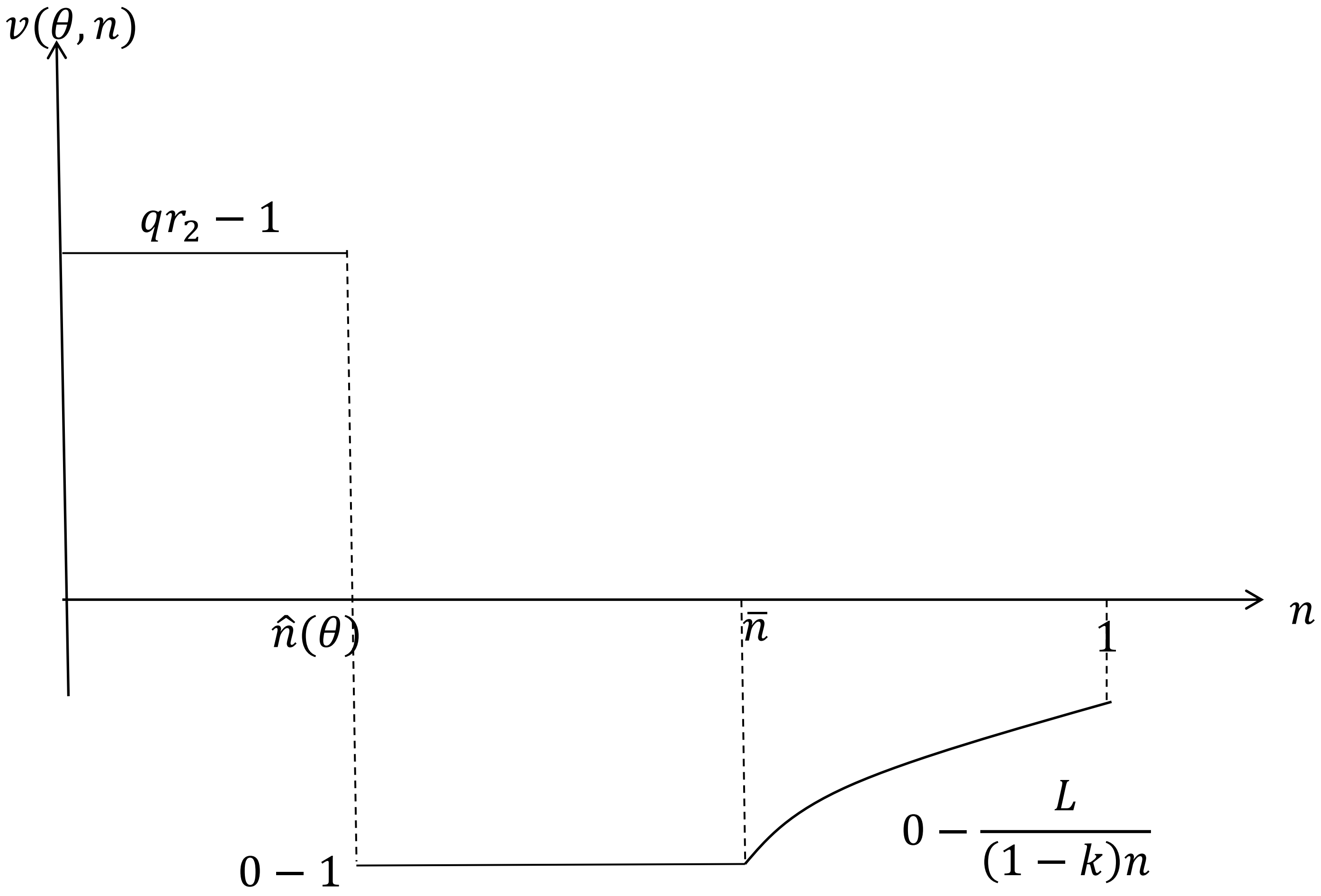


Figure 3: Depositors' payoff differential when $(1-k) > L$. The figure shows a depositor's payoff differential between withdrawing at date 2 and at date 1 as a function of the proportion of depositors withdrawing early, n, when 1-k > L. When $0 \leq n \leq \hat{n}(\theta)$, the bank has enough resources to make the promised repayments at both date 1 and 2. Thus, a depositor expects to receive r_2 with probability q at date 2, and 1 at date 1. When $\hat{n}(\theta) < n \leq \overline{n}$, the bank can still pay 1 at date 1, while fails to repay depositors at date 2. Finally, when $\overline{n} < n \leq 1$, the bank fails to make the promised repayment at both dates. Hence, a depositor receives nothing at date 2, while a share of the bank's liquidation proceeds at date 1.

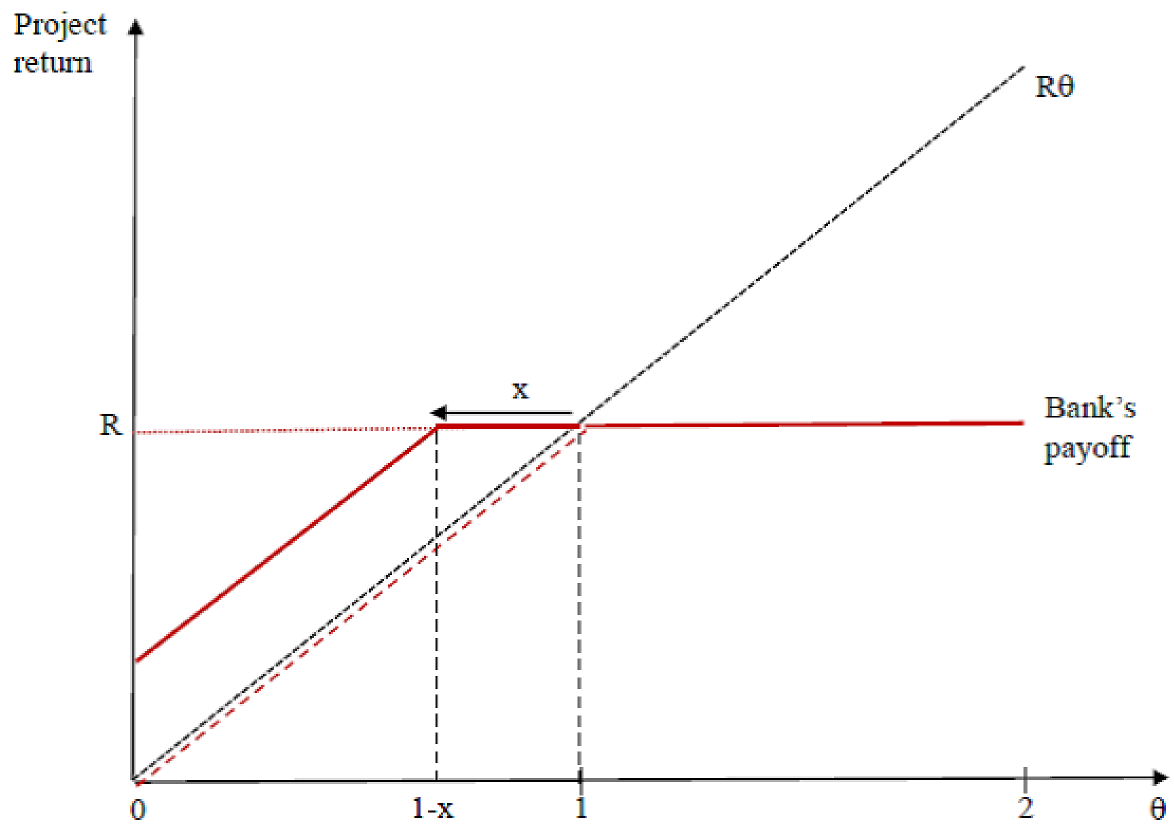
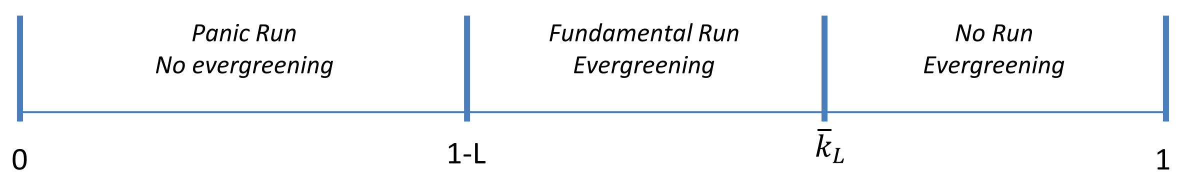


Figure 4: Bank payoff with a first-loss guarantee as a function of the fundamental θ. The figure shows the bank's payoff as a function of θ in the presence of a first-loss guarantee of size x. This is equal to $R(\theta + x)$ for $\theta < 1 - x$ and to R for $\theta \geq 1 - x$.

Economy without guarantees



Economy with a first-loss guarantee with full bankruptcy costs

Figure 5: **Evergreening without and with a first-loss loan guarantee with full bankruptcy costs**. Without guarantees, banks with $k < 1 - L$ are subject to panic runs and do not engage in evergreening, while banks with $k > 1 - L$ do so both when they are subject to fundamental runs for $k\epsilon[1 - L, \bar{k}_L]$, and when they are not subject to runs for $k\epsilon(\bar{k}_L, 1]$. With a first-loss guarantee with full bankruptcy costs, also banks with capital $k\epsilon[\tilde{k}_L, 1 - L)$ start engaging in evergreening.

Acknowledgements

We are grateful to Hans Degryse and to seminar participants at Columbia University, ECB, the Federal Reserve Board, Groening University, John Hopkins University, Halle University, Leuven University, Tuebingen University, UC Davis, University of Luxembourg, FIU, University of Miami, Villanova University, Virtual Finance Theory Seminar, as well as participants at the BIS/CEPR/Deutsche Bundesbank conference on "Evaluating financial regulation:(un)intended effects and new risks" for useful comments.
The views expressed here are the authors' and do not necessarily reflect those of the ECB and the Eurosystem. Elena Carletti acknowledges financial support from Baffi Carefin Centre at Bocconi University.

Elena Carletti
Bocconi University, Milan, Italy; Baffi Carefin, Milan, Italy; Innocenzo Gasparini Institute for Economic Research, Milan, Italy; Centre for Economic Policy Research, London, United Kingdom; email: elena.carletti@unibocconi.it

Agnese Leonello
European Central Bank, Frankfurt am Main, Germany; email: agnese.leonello@ecb.europa.eu

Robert Marquez
University of California, Davis, United States; email: rsmarquez@ucdavis.edu

Postal address 60640 Frankfurt am Main, Germany
Telephone +49 69 1344 0
Website www.ecb.europa.eu

PDF ISBN 978-92-899-5524-9 ISSN 1725-2806 doi:10.2866/546702 QB-AR-23-019-EN-N