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by Oreste Tristani

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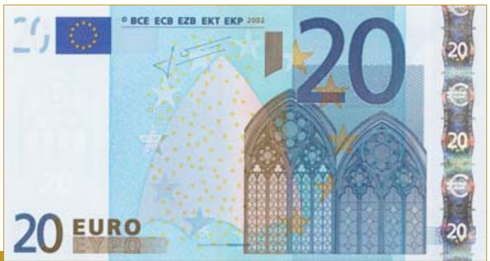

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CONTENTS

Abstract

This paper analyses the determinants of the natural rate of interest in a non-linear model where agents are uncertain over both future technology growth and the future course of monetary policy. I show that the real natural rate can be affected by sizable uncertainty premia, including premia associated with monetary uncertainty. This result is potentially problematic for both the estimation of the natural rate and its use as a policy indicator. Monetary uncertainty can also contribute to amplify the equity premium, and to account for its apparent, positive link with inflation.

Keywords: natural rate of interest, equity premium puzzle, risk-free rate puzzle, robust control, model misspecification.

JEL Classification: E43, G11

Non-technical summary

The long-run equilibrium level of the real interest rate is often referred to as the *natural* interest rate, because it is only affected by structural characteristics of the economy and, notably, it is independent of monetary policy. As a result, one way to define the monetary policy stance is in terms of deviations of the real interest rate from the natural level. If a monetary authority measured the policy stance with reference to an incorrect level of the natural rate, the inflation objective would never be attained.

Correct estimates of the natural rate are therefore useful for central banks. Understanding its determinants is also important, for example to be able to identify in a timely fashion reasons why it could change.

The macroeconomic literature typically relates the natural rate to the time preference rate and to the average rate of productivity growth. Any effects related to risk or a precautionary savings motive and arising from the nonlinearity of economic models are assumed to be negligible. Typically, the literature also assumes that the natural rate can simply be measured as the difference between the average nominal interest rate and average inflation over a sufficiently long sample.

This paper questions these assumptions and studies how the notion of natural rate of interest is affected by model-uncertainty-premia. These premia are associated with households' ignorance of the true probability law governing the stochastic processes of the state variables of the model. More specifically, households are concerned that their perceptions of trend productivity growth and of the future course of monetary policy are incorrect. I allow for differences in the degree of confidence in these two state variables to explore separately the role of "real" and "monetary" uncertainty-premia.

The novel result of the paper is that the (real) natural rate can be significantly affected by premia associated with uncertainty over the future course of monetary policy (i.e. trend money growth). This result arises because households are not certain that their benchmark model of the economy is correct. More specifically, they are not fully confident of their estimates of the trend growth rates of productivity and money supply. They fear that the observed correlation between shocks to productivity and money growth is spurious, i.e. potentially induced by their incorrect assessment of the trend growth rates of both these variables. Uncertainty over the future course of policy can thus be confused with uncertainty over trend productivity growth and, as a result, induce uncertainty premia on real variables.

Care must therefore be taken when drawing policy implications from the natural rate. On the one hand, the same notion of natural rate can become elusive, because it is affected by monetary uncertainty premia which depend on the correlation between monetary and technology shocks. To the extent that this correlation is policy induced, the equilibrium real interest rate is no longer *natural* in the sense of *independent of monetary policy*. On the other hand, estimates of the natural rate may be biased by an inflation premium, if constructed as the difference between the average nominal interest rate and inflation over long periods of time. The bias is unlikely to be huge, but any mismearurement would immediately be reflected in an incorrect policy stance and in undesired inflation volatility.

Finally, the paper shows that the equity premium also incorporates a component due to monetary uncertainty. This feature of the model is useful to account for the link between inflation and the equity premium apparent in US data. Moderate changes in investors' uncertainty over the future course of monetary policy can in fact produce substantial changes in the equity premium. As in previous studies incorporating model uncertainty in investors' behaviour, however, my model can only solve the equity premium puzzle under highly pessimistic assumptions on the risks to the future outlook for consumption growth.

1 Introduction

Following Wicksell (1898), the long-run equilibrium level of the real interest rate is often referred to as the *natural* interest rate, because it is only affected by structural characteristics of the economy and, notably, it is independent of monetary policy. As a result, one way to define the monetary policy stance is in terms of deviations of the real interest rate from the natural level. This is immediately clear if policy is specified as a simple interest rate rule *à la* Taylor (1993), in which the intercept must coincide with the sum of the natural rate and the inflation target.[1] If a monetary authority measured the policy stance with reference to an incorrect level of the natural rate, the inflation target would never be reached.

Correct estimates of the natural rate are therefore useful for central banks. Understanding its determinants is also important, for example to be able to identify in a timely fashion reasons why it could change.

This paper studies how the notion of natural rate of interest is affected by model-uncertainty-premia. These premia are associated with households' ignorance of the true probability law governing the stochastic processes of the state variables of the model. More specifically, households are concerned that their perceptions of trend productivity growth and of the future course of monetary policy are incorrect. I allow for differences in the degree of confidence in these two state variables to explore separately the role of "real" and "monetary" uncertainty-premia.

My main result is that the (real) natural rate can be significantly affected by premia associated with uncertainty over the future course of monetary policy (i.e. trend money growth). At first sight, this finding is surprising, given that money is superneutral in the benchmark model of the economy used by households. However, households are not certain that this model is correct. More specifically, they are not fully confident in their estimates of the trend growth rates of productivity and money supply. They fear that the observed correlation between shocks to productivity and money growth is spurious, i.e. potentially induced by their incorrect assessment of the trend growth rates of both these variables. Uncertainty over the future course of policy can thus be confused with uncertainty over trend productivity growth and, as a result, induce uncertainty premia on real variables.

Care must therefore be taken when using the natural rate to draw policy implications. On the one hand, the same notion of natural rate can become elusive, because it is affected by monetary uncertainty premia which depend on the correlation between monetary and technology shocks. To the extent that this correlation is policy induced, the equilibrium real interest rate is no longer *natural* in the sense

[1] The Taylor-type rules most frequently used in the empirical literature, which include an "interest rate smoothing" motive, also require a correct estimate of the natural interest rate. Variants of the rule in which it is not necessary to specify the natural rate have been analysed by Fuhrer and Moore (1995) and Orphanides and Williams (2002).

of *independent of monetary policy*. On the other hand, estimates of the natural rate may be biased by an inflation premium, if constructed as the difference between the average nominal interest rate and inflation over long periods of time. The bias is unlikely to be huge, but given the important role of the natural rate within Taylor-type rules, any mismeasurement would immediately be reflected in an incorrect policy stance and in undesired inflation volatility.

The paper shows that the equity premium also incorporates a component due to monetary uncertainty. This feature of the model is useful to account for the link between inflation and the equity premium apparent in US data. Moderate changes in investors' uncertainty over the future course of monetary policy can in fact produce substantial changes in the equity premium. As in previous studies incorporating model uncertainty in investors' behavior, however, my model can only solve the equity premium puzzle under highly pessimistic assumptions on the risks to the future outlook for consumption growth.

All results are obtained analytically, in closed form, within a relatively stylized model. Amongst other simplifying assumptions, I postulate that exogenous shocks are serially uncorrelated, so that there is no persistence in the model and the adjustment process is instantaneous. As a result, the natural interest rate is always constant at its long run level and is not subject to cyclical fluctuations. The difference between the long-run notion of natural interest rate emphasized above and the short-run notion used by Woodford (2003) ("equilibrium real rate of return in the case of fully flexible prices", p. 248) is therefore immaterial for my results.

This paper is part of a fast-growing literature exploring various economic implications of model misspecification (also referred to as model uncertainty, or ambiguity), i.e. of a situation where the probability distribution of future events is not known precisely. Building on a version of the Lucas (1978) model (either in discrete or in continuous time), Epstein and Wang (1994) and Chen and Epstein (2002) introduce Knightian uncertainty by allowing for multiple priors, while Hansen, Sargent, and Tallarini (1999), Hansen and Sargent (2001), Anderson, Hansen, and Sargent (2000) and Maenhout (2004) introduce model misspecification and a preference for "robustness". As in these papers, I explore the asset pricing implications of model misspecification. However, my work differs because I use a monetary model and focus on the implications of *monetary* uncertainty for investors' decisions. In so doing, I model monetary policy through a simple rule. A different strand of literature has focused on the normative choice of a robust approach to monetary policy, when the central bank does not know precisely the true model describing the environment – see e.g. Hansen and Sargent (2006), Onatski and Stock (2002), Kasa (2002), Giannoni (2007). Woodford (2005) studies optimal monetary policy when the central bank recognizes that private-sector expectations need not be precisely model-consistent.

The model misspecification approach is also related to the work relaxing the rational expectations hypothesis that agents know the true probability law. Abel (2002) and Cecchetti, Lam, and Mark (2000) use this assumption in order to address the equity premium puzzle. As in these papers, I ultimately derive asset pricing relationships based on a "distorted" probability distribution. However, the choice of the distorted belief is less *ad hoc* in my setting, as it is derived endogenously from the agent's preferences for robustness.

The paper is organized as follows. I motivate and outline the main ingredients of my model in Section 2, where I also discuss the equilibrium asset pricing equation which shapes all risks and uncertainty premia. Section 3 presents results on the natural rate of interest, with particular emphasis on how it is affected by monetary policy uncertainty. Section 4 discusses briefly the equity premium, while a model calibration exercise is presented in Section 5. Section 6 concludes. Technical details can be found in the appendix.

2 Robust consumption in a monetary economy

The macroeconomic literature typically relates the natural rate to the time preference rate and to the average rate of productivity growth. Any effects related to risk or a precautionary savings motive and arising from the nonlinearity of economic models are assumed to be negligible. Typically, the literature also assumes that the natural rate can simply be measured as the difference between the average nominal interest rate and average inflation over a sufficiently long sample.[2]

Since my aim is to relax these assumptions, I wish to construct a model which retains two key features.

First, I want to adopt a general equilibrium framework, to be able to relate the natural rate of interest to taste and other exogenous parameters. I also wish to use a monetary model, to be able to relate the unobservable natural rate of interest to the observable nominal interest rate. I introduce money in the model through the money-in-the-utility-function formulation, as in Sidrauski (1967). As a result, inflation and monetary policy also appear explicitly in the model, albeit in a stylized fashion. More specifically, monetary policy is implemented according to a simple, stochastic money growth rule. This assumption implies that any short-term policy reaction to economic developments is subsumed in the properties of shocks to the rate of growth of money. The correlation of money growth shocks with technology shocks,

[2] I am implicitly assuming that the long-run rate of productivity growth is not subject to permanent shocks. Laubach and Williams (2003) present empirical estimates of the natural rate of interest allowing for permanent productivity shocks.

in particular, could capture the effects of a systematic monetary policy response to deviations of output growth from its trend. For example, a positive correlation may be caused by the tendency of the central bank to neutralize the effects of technology shocks on inflation, so that a negative (i.e. inflationary) technology shock would typically be met by a monetary contraction.

Second, I want to adopt a model of robust decision making which is sufficiently rich to allow me to distinguish between various sources of model uncertainty. I therefore adopt the approach due to Uppal and Wang (2003), which generalizes those in Anderson, Hansen, and Sargent (2000) and Maenhout (2004) allowing one to model separately uncertainty over different state variables of the model. In my model, households are uncertain to different degrees about the trend rate of growth of an exogenous stochastic endowment – which I refer to as technology growth – and about the trend rate of growth of money. Both sources of uncertainty, which are possibly "correlated" in a sense defined below, play an important role in equilibrium.

2.1 A monetary economy

The basic ingredients of the reference model are relatively standard. The model is in continuous time and related to those used by Stulz (1986) and Rebelo and Danyang (1999). Excluding money, the model can be seen as a continuous time equivalent of the Lucas (1978) model.

The representative household chooses an optimal consumption/investment plan, which requires selecting a consumption rate and the portfolio shares of a number of assets in order to maximize the intertemporal utility function

$$\int_0^{\infty} e^{-\delta t} \frac{\left[u\left(c_t, m_t\right)\right]^{1-\gamma}}{1-\gamma} dt \qquad \gamma \neq 1 \tag{1}$$

subject to a standard wealth accumulation constraint and to a concern for model misspecification discussed in section 2.2. In equation (1), $\delta > 0$ is the rate of time preference, $\gamma > 0$ is the coefficient of relative risk aversion, c_t is consumption and m_t represents real money balances (henceforth, I drop all time subscripts to simplify the notation). Current utility, in turn, takes the CES form

$$u\left(c, m\right) = \left(\alpha^{\frac{1}{\varepsilon}} c^{\frac{\varepsilon-1}{\varepsilon}} + \left(1-\alpha\right)^{\frac{1}{\varepsilon}} m^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{2}$$

where $0 < \alpha \leqslant 1$ is a constant and ε is the elasticity of intratemporal substitution between consumption and real money balances.[3] The investment opportunity set is kept simple, but it is sufficiently rich to analyze the equilibrium variables of interest. Thus, the household can invest its real wealth, w, in: claims to the future endowment stream, i.e. equity s yielding a risky instantaneous return $\mu_s \mathrm{d}t + \sigma_s \mathrm{d}z_s$, where $\mathrm{d}z_s$ is a standard Brownian motion; real balances, yielding zero nominal return; a real bond, b, in zero net supply, yielding a safe instantaneous real rate $r\mathrm{d}t$; and a nominal bond, B, also in zero net supply, yielding a nominal interest rate $R\mathrm{d}t$. Since money is injected in the economy through stochastic nominal transfers, I also introduce an additional asset x, which represents the expected present discounted value of future money transfers and therefore ensures the household from future transfer shocks. The portfolio shares in the 5 assets available in the economy are denoted by ω_i, for $i = s, M, B, x, b$.

All the asset returns will be determined in equilibrium, together with the inflation rate. The only exogenous processes driving the economy are those specifying the rate of growth of the endowment process, y, and the supply of nominal money, M. To maintain tractability, both processes are assumed to be geometric Brownian motions

$$\frac{\mathrm{d}y}{y} = \mu_y \, \mathrm{d}t + \sigma_y \, \mathrm{d}z_y \tag{3}$$

$$\frac{\mathrm{d}M}{M} = \mu_M \, \mathrm{d}t + \sigma_M \, \mathrm{d}z_M \tag{4}$$

where $\mathrm{d}z_y$ and $\mathrm{d}z_M$ are standard, possibly correlated, Brownian motion processes (with correlation coefficient ρ_{My}), and μ_y, μ_M, σ_y and σ_M are constant, exogenous parameters.

As already discussed above, any systematic policy reaction to macroeconomic developments is subsumed in the covariance between monetary and technology shocks. A policy rule such that the covariance is zero would be "passive", in the traditional sense associated with Milton Friedman.

The aforementioned ingredients of the model are standard. The main aspect in which I depart from previous analyses is to account for model misspecification.

[3] The elasticity of intra-temporal substitution does not play any significant role in the model. More precisely, all our propositions would remain unchanged if we selected the simpler Cobb-Douglas form $u(c, m) = c^{\alpha} m^{(1-\alpha)}$, where $\varepsilon = 1$. We adopt the more general CES formulation to be able to investigate the case of $\gamma < 1$ in the numerical simulations without having to assume at the same time that consumption and money are substitutes (i.e. that the cross derivative $u_{cm} < 0$).

2.2 Model misspecification

The simple model outlined above is well-known to be ill-suited to account for the observed behavior of economic agents towards risk when solved under rational expectations. In order to improve its empirical performance, I follow the suggestion of the recent literature on robust control and use a different assumption for individuals' behavior. More specifically, I assume that households are uncertain about the exact distributions of the rate of growth of money and the rate of growth of productivity. They therefore use the model as a benchmark and try to devise consumption and investment strategies which can yield "good" outcomes if the model is misspecified. The underlying idea is that the differences between alternative models and the reference model are difficult to detect empirically, but consequential for households' decisions.

The endowment and money growth processes described in equations (3) and (4) represent the *status quo* knowledge of the representative household and they are defined under a certain probability measure P, the reference probability or "reference model." If P were known with full confidence, there would be no uncertainty, or "ambiguity," as to the reference model. Since households are not fully confident on P, they entertain the possibility that different models may be true, where the different models are indexed by probabilities Q^{ξ} induced by perturbations ξ's of the state vector. Q^{ξ} is then defined by $dQ^{\xi} = \xi dP$ and under the Q^{ξ} measure the law of motion followed by the state vector is simply modified by a drift-adjustment term.

Intuitively, households take as a benchmark a model where the trend rates of technology and money growth are $\mu_y dt$ and $\mu_M dt$, respectively. When taking decisions, however, they entertain the possibility that the actual drifts will be different from the benchmark values, consistently with the idea that it is difficult to estimate exactly the drift of the processes followed by the state variables.

The magnitude of the deviations from the reference model will depend on a matrix Φ that indexes the household's confidence on the law of motion of the state vector. When the elements in Φ are very large, the household is extremely confident on the reference model and it will not bother taking alternative models into account. At the other extreme, when the elements in Φ are very small, the household is so uncertain on the reference model that it will act based on a worst-case scenario. In all intermediate cases, the household balances its knowledge of the reference model and its concern about model misspecification.

Following Uppal and Wang (2003), the matrix Φ allows for different degrees of confidence over the various elements of the state vector. More specifically

$$\Phi = \phi_{My} \begin{pmatrix} \sigma_y^2 y^2 & \sigma_{My} My \\ \sigma_{My} My & \sigma_M^2 M^2 \end{pmatrix} + 2\phi_y \begin{pmatrix} \sigma_y^2 y^2 & 0 \\ 0 & 0 \end{pmatrix} + 2\phi_M \begin{pmatrix} 0 & 0 \\ 0 & \sigma_M^2 M^2 \end{pmatrix} \quad (5)$$

so that ϕ_{My} indexes households' knowledge of the joint distribution of the state vector, while ϕ_y and ϕ_M denote the confidence on, respectively, the marginal distributions of the technology and money growth process ($\phi_M, \phi_y, \phi_{My} \geqslant 0$).

This specification allows me to analyze independently different levels of confidence in the trend growth rate of technology $\mu_y \mathrm{d}t$ and of money supply $\mu_M \mathrm{d}t$. For example, households would confidently rule out the possibility that $\mu_y \mathrm{d}t$ reaches double-digit levels, because it is constrained by the arrival of technological innovations. For money growth, however, no such technological constraint exists and one may want to take into account the possibility of very high inflation scenarios. The relatively low uncertainty over the marginal distribution of the endowment process would be reflected in a relatively large ϕ_y element of the Φ matrix. For the marginal distribution of the money supply process, however, the household may want to entertain the possibility of a much smaller ϕ_M element.

2.3 An asset pricing equation robust to real and monetary uncertainty

Definition 1 *The robust equilibrium is characterized by consumption and money demand rules, portfolio shares and prices such that: (i) the households solve the Bellman equation subject to the appropriate boundary condition (see the appendix); (ii) markets clear continuously, i.e.* $c = y$, $m^d = M^s/P$, $\omega_B = \omega_x = \omega_b = 0$, $\omega_s + \omega_M + \omega_x = 1$.

The appendix shows that the asset pricing equation of the model is

$$\mu_i = r + \gamma\, \rho_{iw}\, \sigma_i\, \sigma_w + \left(\frac{1}{\Phi_{My}} + \frac{1}{\Phi_y} \right) \rho_{iy}\, \sigma_i\, \sigma_y + \frac{1}{\Phi_M}\, \rho_{My}\, \sigma_y\, \rho_{iM}\, \sigma_i \quad (6)$$

where

$$
\begin{aligned}
\Phi_{My} &\equiv \frac{2\phi_M\phi_y + \phi_{My}\left(\phi_M + \phi_y + \frac{1}{2}\phi_{My}\left(1-\rho_{My}^2\right)\right)}{\frac{1}{2}\phi_{My}\left(1-\rho_{My}^2\right)} \\
\Phi_y &\equiv 2\phi_y + \frac{\phi_{My}}{\phi_M}\left(\phi_M + \phi_y + \frac{1}{2}\phi_{My}\left(1-\rho_{My}^2\right)\right) \\
\Phi_M &\equiv 2\phi_M + \frac{\phi_{My}}{\phi_y}\left(\phi_M + \phi_y + \frac{1}{2}\phi_{My}\left(1-\rho_{My}^2\right)\right)
\end{aligned}
$$

Note that $\Phi_{My}, \Phi_y, \Phi_M \geq 0$ because all ϕ's are positive by construction and $0 \leq \rho_{My} \leq 1$. In the rational expectations equilibrium, $\Phi_{My} = \Phi_y = \Phi_M = \infty$ and only the standard risk-premium $\gamma\rho_{iw}\sigma_i\sigma_w$ would appear in equation (6). Under the assumption of a unique source of model misspecification, Φ_{My} simplifies to $\Phi_{My} = \phi$ and $\Phi_M, \Phi_y \to \infty$.

Equation (6) shows that the two sources of model uncertainty existing in the model – on technology and money growth – translate into two sorts of uncertainty premia. The premium attached to technological uncertainty is proportional to the covariance term $\rho_{iy}\sigma_i\sigma_y$. It insures households against a scenario in which technology growth, hence the rate of growth of consumption, turns out to be lower than the level predicted by the reference model. Households would like to hold assets which yield a high return if technology growth is lower than in the reference model, i.e. assets whose price correlation with technology shocks, ρ_{iy}, is negative. Thus, assets that covary positively with technology growth have to pay a model uncertainty premium.

The premium attached to monetary uncertainty, namely the term $\rho_{My}\sigma_y\rho_{iM}\sigma_i$, is the novel result of my model. The monetary uncertainty premium has to do with the active nature of the policy rule when $\rho_{My} \neq 0$. A monetary policy shock, which could represent the systematic policy reaction to a technology shock in the reference model, is interpreted as a signal of model misspecification in an uncertain environment, i.e. a signal that the reference trend growth rates of both money and productivity are incorrect.

The monetary uncertainty premium is increasing in the correlation ρ_{My}. If the correlation is positive, the possibility that trend money growth is lower than in the reference model is bad news, in the sense that it may imply that technology growth is also lower than in the reference model. Assets whose prices covary positively with monetary shocks are therefore subject to an uncertainty premium, as they yield a low return exactly when consumption growth is low, if the model is misspecified.

If $\rho_{My} < 0$, assets whose price is positively correlated with monetary shocks will trade at a discount, because they provide insurance against the risk of model misspecification. Only when $\rho_{My} = 0$ – i.e. when policy is passive – does the amount of monetary uncertainty vanish (regardless of the correlation between asset prices and money growth). When monetary and technology shocks are uncorrelated in the reference model, there is in fact no reason for households to believe that misspecification of money growth may be a signal of misspecification of the model of technology growth.[4]

The monetary uncertainty premium is independent of the standard deviation of monetary shocks. Their size is irrelevant because they only matter for asset pricing for the indirect information they convey on technology shocks.

The compounded coefficients Φ_{My}, Φ_y, Φ_M show how households' confidence over the distribution of the technology and money growth processes translates into required returns on assets. The premium attached to monetary (technological) uncertainty is decreasing in the ambiguity ϕ_M (ϕ_y) of the reference model of monetary (technological) growth. At the same time, when uncertainties are correlated ($\phi_{My} \neq 0$), the monetary (technological) uncertainty premium is increasing in ϕ_y (ϕ_M). The intuition is that, for given ϕ_M and ϕ_{My}, an increase in ϕ_y means that the existing uncertainty over the correlation between technology and monetary shocks can be attributed to a larger degree to uncertainty over the marginal distribution of monetary shocks. In this respect, the ambiguity of the reference model of the monetary policy rule increases.

This cross-effect disappears when there are no links between technological and monetary uncertainty, i.e. when $\phi_{My} = 0$, so that Φ_y and Φ_M are independent of ϕ_M and ϕ_y, respectively. In this case, the ambiguity parameters simplify to $\Phi_y|_{\phi_{My}\to 0} = 2\phi_y$ and $\Phi_M|_{\phi_{My}\to 0} = 2\phi_M$.

[4] Uncertainty on the monetary policy rule would continue to affect optimal portfolio shares also in this special case. The asset pricing equation would simplify to $\mu_i = r + \gamma\,\rho_{iw}\,\sigma_i\,\sigma_w + \frac{1}{\phi_{My}+2\,\phi_y}\,\rho_{iy}\,\sigma_i\,\sigma_y$ and the degree of confidence in the monetary policy rule would affect optimal portfolio shares through the uncertainty premium on technology shocks. More precisely, the smaller ϕ_{My}, i.e. the larger the correlation between uncertainties over technology and money growth, the larger the risk that the reference model may be overestimating the rate of technology growth and, in turn, the larger the uncertainty premium required because of potential misspecification of the technology process.

3 Monetary uncertainty and the natural rate of interest

In this section, I explore the implications of the model for the natural rate of interest, i.e. the long-run equilibrium level of the real interest rate.[5]

In the deterministic steady-state of macroeconomic applications, this notion is straightforward to evaluate. In a stochastic equilibrium, however, there is a degree of ambiguity as to the appropriate definition of a *real* rate of return. One possibility is to use the real return on a nominal bond, i.e. the equilibrium nominal interest rate net of expected inflation. An alternative possibility is to rely on the return on a safe real bond. The difference is that the former definition will in general include an inflation risk premium, even in rational expectations equilibria. This is an undesirable feature, since it would imply that the natural rate could no longer be viewed as an indicator of exogenous inflationary pressures.

To ensure that the natural rate is independent of monetary policy in a rational expectations equilibrium, it is preferable to define it as the "equilibrium rate of return on a (safe) real bond." This definition corresponds to that of risk-free rate in the finance literature. In the rest of the paper, the expected real return on a nominal bond will simply be referred to as "real rate".

Proposition 1 *In the robust equilibrium, the natural interest rate (or risk-free rate) is*

$$r = \delta + \gamma\mu_y - (1+\gamma)\left(\gamma + \frac{1}{\Phi_{My}} + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right)\frac{\sigma_y^2}{2} \tag{7}$$

The natural rate is decreasing in the level of ambiguity of the reference model and, in particular, in the level of uncertainty over trend money growth.

This proposition shows that the impact of model misspecification on the natural interest rate is always negative. A fall in households' confidence on the accuracy of the reference model has the same effect as an increase of "technological risk," σ_y. Both scenarios imply a less predictable future evolution of the endowment, thus an increase in precautionary savings and a fall in the natural rate.

To compare this result to the standard rational expectations equilibrium, it is

[5] Given that there is no nominal rigidity in the model, the natural rate could be equivalently defined, following Woodford (2003), as "the equilibrium real rate of return in case of fully flexible prices"

useful to rewrite equation (7) as

$$r = \delta + \gamma\mu_{\widehat{y}} - \gamma^2 \frac{\sigma_y^2}{2} - (1+\gamma)\left(\frac{1}{\Phi_{My}} + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right)\frac{\sigma_y^2}{2} \tag{7b}$$

where $\mu_{\widehat{y}}$ is defined as $\mu_{\widehat{y}} \equiv \mathrm{E}\left[\mathrm{d}\ln y\right]/\mathrm{d}t$ and I have used the equivalence $\mathrm{d}\ln y = \mathrm{d}y/y - \left(\sigma_y^2/2\right)\mathrm{d}t$. I would obtain equation (7b) if I assumed at the outset the process $\mathrm{d}\ln y = \mu_{\widehat{y}}\mathrm{d}t + \sigma_y \mathrm{d}z_y$, rather than that in equation (3), for technology shocks. This slight change facilitates the comparison with some of the literature developed in discrete time. In the absence of model misspecification, i.e. when $\Phi_{My}, \Phi_y, \Phi_M \to \infty$, the model yields the conventional result $r = \delta + \gamma\,\mu_{\widehat{y}} - \gamma^2\,\sigma_y^2/2$ – see e.g. equation (15) in Campbell (2003).

The contribution of this paper is to highlight that the *real* natural rate can also be affected by uncertainty over a *nominal* variable, namely trend money growth. Once again, the correlation between monetary and technology shocks, ρ_{My}, plays a key role. Even if the marginal distribution of the technology shocks were known with full confidence, the joint distribution of monetary and technology shocks would remain ambiguous. The possibility of misspecification of the trend rate of productivity growth would call for an increase of precautionary savings and, as a result, it would cause a fall in the natural rate.

The presence of a monetary uncertainty premium amongst the determinants of the natural real rate is consequential for its relationship with monetary policy. In the robust equilibrium, the natural rate is affected by the correlation between real and monetary shocks. To the extent that the correlation is policy-induced, the natural rate is no longer independent of monetary policy.

This result is potentially problematic for the use of the natural rate as a policy indicator. The usefulness of the natural rate is to signal the (real) interest rate level which is consistent with a "neutral" – i.e. neither expansionary nor contractionary – policy stance. Under a rational expectations equilibrium, a central bank would be able to take this level as given and define its policy in terms of how quickly it wishes to return to the neutral stance after any given shock. Under model misspecification, this two-step approach is not feasible. A systematically more aggressive reaction to inflationary episodes arising from technology shocks, for example, would increase the correlation ρ_{My} and cause a fall in the natural rate. The natural rate would thus no longer represent a benchmark of neutral policy stance which can be defined independently of the systematic characteristics of monetary policy.

A more standard implication of the existence of an inflation premium is to open a wedge between the natural rate and the real rate, hence to complicate the estimation of the natural rate as the average difference between nominal rates and expected inflation over long time-periods. This issue is discussed next.

3.1 Monetary uncertainty and the real rate

To investigate how different the natural rate is from the real rate – i.e. the nominal rate minus expected inflation – I first need to derive the equilibrium values of the nominal variables of the model. In order to highlight the effect of inflation on the nominal interest rate, I assume that the central bank steers the money supply in order to achieve an inflation objective π^* in expected terms.[6] It follows that:[7]

Proposition 2 *In the robust equilibrium, the trend rate of growth of money which supports the exogenous inflation target* $\mathrm{E}[\pi^*]$ *is* $\mu_M^* = \mathrm{E}[\pi^*] + \mu_y + \sigma_M^2 - \rho_{My}\,\sigma_M\,\sigma_y$*. Given this equation, the nominal interest rate can be written as*

$$R = r + \mathrm{E}[\pi^*] + IP \tag{8}$$

where r *is the natural rate defined in equation (7) and the inflation risk-plus-uncertainty-premium on nominal bonds,* IP*, is given by*

$$IP = \left(\gamma + \frac{1}{\Phi_{My}} + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right)\sigma_y^2 - \left(\gamma + \frac{1}{\Phi_{My}} + \frac{1}{\Phi_y} + \frac{1}{\Phi_M}\right)\rho_{My}\,\sigma_M\,\sigma_y \tag{9}$$

Definition 2 *The ex-ante real rate is defined as the equilibrium real return on the nominal bond, or* $r^e \equiv R - \mathrm{E}[\pi^*]$*. The ex-ante real rate is equal to the sum of the natural rate and the inflation premium, or* $r^e = r + IP$*.*

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[6] Monetary and technological shocks make it impossible to attain the inflation objective at all times. Inflation volatility is typically non-zero in the model. It disappears, so that inflation becomes deterministic, only if $\rho_{My} = 1$ and $\sigma_M = \sigma_y$. This can be appreciated from the expression for equilibrium inflation

$$\frac{dP}{P} = \left(\mu_M - \mu_y + \sigma_y^2 - \rho_{My}\sigma_M\sigma_y\right)dt + \sigma_M dz_M - \sigma_y dz_y$$

[7] We could specify the inflation target directly in terms of the rate of growth of prices P. For bond holders, however, the price level matters as a deflator of the price of nominal bonds, whose real value is B/P. Given the nonlinearity of the model, an inflation target specified in terms of the rate of growth of P would imply that some convexity terms would play a role on the equilibrium interest rate. In order to highlight the "pure" influence of risk and uncertainty aversion on the equilibrium interest rate, we therefore specify the inflation target in terms of the price deflator $1/P$, i.e. $E[\pi^*] \equiv -(1/P)\,E[d(1/P)]/dt$.

As already mentioned, the real rate is affected by the standard deviation of monetary shocks, through the inflation premium. Consequently, it is not independent of the monetary policy rule followed by the central bank. Under a Taylor-type rule, for example, the equilibrium value of the policy interest rate would be affected by the parameters defining the short-term features of the rule. Different rules tolerating various degrees of short-term inflation volatility would be consistent with different equilibrium values of the real rate.

The inflation premium IP reflects the fact that, in this model, surprise inflation arises for two reasons: negative technology shocks and positive money growth shocks. The former are clearly undesirable, because they are perfectly correlated with (or actually identical to) shocks to real wealth. Thus, they command positive risk and uncertainty premia, represented by the component $\left(\gamma + 1/\Phi_{My} + 1/\Phi_y + \rho_{My}^2/\Phi_M\right)\sigma_y^2$. As to inflationary episodes due to shocks to money growth, these only command a premium insofar as they are negatively correlated with technology shocks – that is, insofar as they tend to be accompanied by negative shocks to wealth. If instead shocks to money growth are positively correlated with technology shocks, then the surprisingly low real return on nominal bonds due to an unexpected monetary expansion tends to be compensated by a positive endowment shock. When $\rho_{My} > 0$, nominal bonds are a hedge against negative shocks to wealth and therefore pay a negative premium. This is the component $-\left(\gamma + 1/\Phi_{My} + 1/\Phi_y + 1/\Phi_M\right)\rho_{My}\sigma_M\sigma_y$.[8]

The overall sign of the inflation premium is ambiguous, as it depends on whether the premium required because of real shocks is larger or smaller than the discount accepted because of the possibility of monetary shocks. The inflation premium can in fact be split into two components: $\left(\gamma + 1/\Phi_{My} + 1/\Phi_y\right)\left(\sigma_y^2 - \rho_{My}\sigma_M\sigma_y\right)$ and $\left(1/\Phi_M\right)\left(1/\sigma_M\right)\rho_{My}\sigma_y\left(\rho_{My}\sigma_M\sigma_y - \sigma_M^2\right)$. In empirical calibrations where the standard deviation of technology shocks is approximated by that of consumption growth, the covariance between consumption and monetary shocks is positive but typically smaller than the variance of either consumption or money growth. Hence, the first component of the inflation premium tends to be positive, while the second one is often negative.

Finally, the size of the inflation premium relative to the equity premium depends on the sign of the covariance between money and technology shocks. When this is

[8] The inflation premium discussed here is different from the term premium typically analysed in term structure models. The latter is normally a relative premium of long-term over short-term bonds. Here, the inflation premium is an absolute premium paid by short-term bonds over and above the risk-free rate.

positive, as is often the case in the empirical evidence, the equity premium is larger – which appears to be intuitively appealing. Ultimately, the sign and size of the inflation premium must be determined empirically.

4 Monetary uncertainty and the equity premium puzzle

Compared to analogous expressions obtained under rational expectations, equation (7) includes additional free parameters. I therefore need to impose discipline on the calibration of my model. One way in which I do this is to force the model to match the data on the equity premium (an additional way is to check how pessimistic is the growth scenario implicit in the robust equilibrium – see section 5).

4.1 The equity premium

Proposition 3 *The equity premium, $EP \equiv \mu_S - r$, is given by*

$$EP = \left(\gamma + \frac{1}{\Phi_{My}} + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M} \right) \sigma_y^2 \tag{10}$$

The equity premium is increasing in the uncertainty over the distribution of both technology and monetary shocks.

This result encompasses both the case of a single source of model misspecification studied by Anderson, Hansen, and Sargent (2000) and Maenhout (2004), in which the equity premium collapses to $EP = \left(\gamma + \frac{1}{\phi}\right) \sigma_y^2$, and the case of a standard rational expectations equilibrium, where $EP = \gamma\sigma_y^2$.

Once again, the novel feature of Proposition 3 concerns the role of uncertainty over the monetary policy rule. Changes in households' confidence in the monetary policy rule, which imply changes in confidence in the equilibrium inflation level consistent with the reference model, affect the equity premium. More specifically, an increase in uncertainty over trend money growth, hence over equilibrium inflation, will amplify the equity premium, as $\partial EP/\partial\phi_M = -2\left(\rho_{My}^2/\Phi_M^2\right)\sigma_y^2 < 0$. Note also that the size of this effect is larger when technological uncertainty is greater, as $\partial^2 EP/\partial\phi_M\partial\phi_y = -4\left(1/\Phi_{My} + 1/\Phi_y\right)\left(\rho_{My}^2/\Phi_M^2\right)\left(\phi_{My}/\phi_y\right)\sigma_y^2 < 0$. These results are consistent with the idea that the equity premium may change when monetary institutions change, to the extent that the change affects households' confidence in the policy rule of the reference model. The recent stronger focus on price stability by most central banks should, for example, have resulted in a reduction of the level of ambiguity over long-term money growth, thus a fall in Φ_M and in the relative equity premium.

These results can also be seen as consistent with the evidence presented in Blanchard (1993) on historical movements in the US equity premium. Amongst other findings, Blanchard (1993) argues that movements in the equity premium around a long-term trend appear to be correlated with movements in inflation. The correlation is especially clear in the seventies, when the sharp increase in inflation is associated with a higher equity premium, and in the eighties, when a lower premium is associated with the fall in inflation. Blanchard (1993) also argues that "identifying the reason why inflation affects the premium is even more [difficult]" (p. 105). One explanation mentioned in that paper is related to the possibility that investors suffer from money illusion, as argued by Modigliani and Cohn (1979).

My alternative explanation has the advantage of being consistent with the postulates of (robustly) rational decision making. The seventies and eighties could thus be interpreted as different regimes in terms of confidence in the anti-inflationary attitudes of central banks. The high level of uncertainty over trend money growth, which probably characterized the seventies, should indeed be associated with a higher equity premium, as registered by Blanchard (1993). Conversely, the higher confidence in the central banks' anti-inflationary determination during the eighties could explain the fall of the equity premium in those years.

4.2 The relative equity premium

In standard calibrations of the equity premium – starting from Mehra and Prescott (1985) – the risk-free rate is approximated by the average over a long time-period of the ex-ante real rate. Once I acknowledge the conceptual role of inflation premia in driving a wedge between the risk-free rate and the natural nominal rate, the standard calibration procedure is no longer valid – see also Labadie (1989).

I therefore define the "observed" equity premium as the difference between the expected (average) return on equity μ_s and the ex-ante real rate r^e. This is a *relative* premium, as it describes the size of the equity premium over and above the inflation premium. From equation (9) and (10), the relative equity premium can be obtained immediately, since $REP \equiv \mu_s - r^e = \mu_s - r - (r^e - r) = EP - IP$. It is therefore possible to establish the following

Proposition 4 *The "relative equity premium," defined as the expected real return* *paid by equity over the expected real return on nominal bonds, is given by*

$$REP = \left(\gamma + \frac{1}{\Phi_{My}} + \frac{1}{\Phi_y} + \frac{1}{\Phi_M}\right) \rho_{My}\, \sigma_M\, \sigma_y \tag{11}$$

From the conceptual viewpoint, the difference between the equity premium proper and the relative equity premium is stark. Most notably, monetary shocks matter for the relative equity premium, while they are irrelevant for the equity premium proper. The relative premium is increasing in the volatility (i.e. standard deviation) of money growth shocks, when the correlation ρ_{My} is positive. It could also be negative when $\rho_{My} < 0$. The equity premium proper is instead always positive and it is not directly affected by monetary developments.

From a quantitative viewpoint, equation (11) deepens the equity premium puzzle in two ways. First, the covariance between shocks to money and consumption growth – the variables often used in calibrations of the equity premium – is often smaller than the variance of consumption growth. This will typically imply that a larger risk-aversion parameter is necessary to generate the ex-post averages of the equity premium.

A second way in which equation (11) deepens the equity premium puzzle is by showing that the same (technological) factors that produce the equity premium in a simple rational expectations model will also tend to generate an inflation premium on nominal bonds. In this model, the latter component of the inflation premium (the first in equation (9)) is exactly identical to the equity premium: thus, the two components cancel each other out when the relative equity premium is constructed as the difference between the equity premium and the inflation premium. Such a stark result is, obviously, model dependent. In more general models, however, one would also expect high equity premia to be associated with high inflation premia, when both premia are linked to shocks that depress consumption and boost inflation at the same time. This will typically be the case for technological shocks.

I have already discussed above the ability of the model to generate an increase in the equity premium proper during a period of high inflation, which is consistent with the evidence presented in Blanchard (1993). My model can potentially account for large increases in the relative equity premium at times of high inflation, if the latter also causes a fall in households' confidence in the future course of monetary policy – that is, a fall in ϕ_M. Using equilibrium conditions, for realistic values of ρ_{My}, σ_y and σ_M (that is, if $\rho_{My} > 0$ and $\rho_{My}\sigma_y < \sigma_M$), $\partial REP/\partial\phi_M < 0$, $\partial r/\partial\phi_M > 0$

and $\partial r^e/\partial\phi_M > 0$. Hence, an increase in monetary policy uncertainty, i.e. a fall in ϕ_M, leads unambiguously to an increases in the relative equity premium. The size of the increase can be large. For example, focusing on the $\phi_{My} = 0$ case explored in the main calibration, the derivative of the relative equity premium simplifies to $\partial REP/\partial\phi_M = -\left(\frac{1}{2}/\phi_M^2\right)\rho_{My}\,\sigma_M\,\sigma_y$, which can become infinitely large as ϕ_M becomes smaller and smaller.

5 A simple numerical calibration

This section evaluates the quantitative implications of the model. The calibration is mainly based on US data, for which longer time-series are available. For comparison purposes, I also present some tentative results related to the euro area.

The standard preference parameters are calibrated using values consistent with the macroeconomic literature. More specifically, for the time preference parameter, δ, I use two values consistent with the DSGE literature, 3% and 4% – see e.g. Christiano, Eichenbaum, and Evans (2005) and Smets and Wouters (2003). For the coefficient of relative risk aversion, γ, the simulations are based on the values of 0.5 and 6. The first value is in the very low range: it is consistent with a very small degree of risk aversion, but implies an elasticity of intertemporal substitution which is possibly too large. The second value is on the high side, though not as large as the largest considered by Mehra and Prescott (1985) ($\gamma = 10$).

The parameters which determine how extreme is the worst case scenario considered by investors are less easy to pin down. Their values are therefore selected so as to yield, if possible, the "observed" value of the ex-ante real rate and the equity premium. I then investigate the implications of the simulations for the natural rate and the other premia. I also check how pessimistic is the worst case scenario in terms of the perturbed rate of growth of consumption. In the worst case scenario, this is adjusted downwards by the amount $\left(1/\Phi_{My} + 1/\Phi_y + \rho_{My}^2/\Phi_M\right)\sigma_y^2$, so that the worst case scenario for growth becomes $WCG = \mu_y - \left(1/\Phi_{My} + 1/\Phi_y + \rho_{My}^2/\Phi_M\right)\sigma_y^2$. This quantity will be monitored in the simulations below to gauge whether the assumptions on the ambiguity parameters are plausible.

For the US, the moments of consumption and money growth (i.e. μ_y, σ_y, σ_M and ρ_{My}) are calibrated from the series of quarterly consumption and yield data collected by Campbell (1999) and from (seasonally adjusted) data on money (M2) available from the Federal Reserve Board.[9] Since the latter dataset is only available from the beginning of 1959, the sample period used in the calibration is 1959Q1-1999Q4. The sample standard deviations of the rates of growth of money M2 and real consumption, and their correlation coefficient are $\sigma_M = 1.81$, $\sigma_y = 0.95$ and $\rho_{My} = 0.25$, respectively. For the euro area, I use the data constructed for the ECB's area wide model – see Fagan, Henry, and Mestre (2005) – complemented by ECB data for money aggregates.

In my US sample, the average premium on equity is broadly consistent with the conventional wisdom. The average excess return on equity is in fact 5.2 percent, while the ex-ante real interest rate is equal to 1.5 percent. A recent stream of research from Blanchard (1993) to Fama and French (2002), however, has argued that the post-WWII premium could be much lower than previously thought. The largest estimate of the equity premium from 1951 to 2000 in Fama and French (2002), for example, is only 4.3 percent (and it can drop as low as 2.6 percent). In my calibration, I use the sample value as my main benchmark. The 4.3 percent value suggested by Fama and French (2002) is used to gauge some suggestive evidence on possible recent evolution of the natural rate.

5.1 The baseline calibration

As a benchmark for comparison, Table 1 calibrates the model for the cases of no ambiguity ($\phi_y = \phi_{My} = \phi_M = \infty$) and a single source of ambiguity ($\Phi_{My} = \phi$, $\Phi_M, \Phi_y = \infty$). The table reports the equilibrium values of the natural rate, the ex-ante real rate, the relative equity premium, the equity premium proper, the inflation premium and, finally, the worst case scenario for consumption growth, all in percentage points.

For each of the two values of δ taken into account, the first row shows the value of γ necessary to generate the observed average real rate of 1.5 percent. The implied coefficient of risk aversion is implausibly high, which reiterates the evidence of a risk-free rate puzzle. As argued in section 4, looking at the relative equity premium deepens the equity premium puzzle. The exceedingly large risk aversion parameter necessary to match the average real rate does generate a large equity premium proper, but is only capable of producing a relative premium of 2 percent.

[9] Money stock data are from Table 1 of the Federal Reserve Board's "Money Stock Measures" H.6, available from http://www.federalreserve.gov/releases/h6/hist. The results do not change significantly using the moments of M1.

The other rows in the table are based on values of γ of 0.5 or 6 and selecting ϕ so as to match the observed ex-ante real rate. The results show that, consistently with Maenhout (2004), this model is capable of generating a high equity premium, but only at the cost of a overly pessimistic assumption for the worst case scenario. When the risk aversion parameter is equal to 6, the worst case scenario is such that consumption growth is approximately -3 percent. For the lower risk aversion parameter, however, the worst case scenario jumps to -15 percent, certainly an implausible assumption.

The main calibration results are shown in Table 2, which is based on the $\phi_{My} = 0$ assumption throughout. This assumption reflects the hypothesis that the uncertainties over technology and money growth are not correlated, i.e. that people understand that the two rates of growth have different determinants: technological for μ_y, institutional (monetary policy) for μ_M.

In the baseline calibration, reported in the first half of the table, the other parameters, ϕ_M and ϕ_y, are chosen to jointly match the sample mean of the ex-ante real rate and the relative equity premium. In the second half of the table, discussed in the next section, I perform a comparative static exercise aimed to capture a "monetary instability" scenario.

The baseline calibration confirms that the model can indeed generate the observed relative equity premium, though only assuming a relatively high degree of pessimism of the representative household. The worst-case scenario for consumption growth entertained by the household is never higher than -2 percent. Less pessimistic assumptions are only possible for much larger values of γ, which however also yield implausibly high values for the inflation premium.

The slightly improved performance of my model compared to the case of a single source of model misspecification arises from the contribution of monetary uncertainty. This helps to boost the relative equity premium without producing large effects on the degree of pessimism on consumption growth (the effect on the latter is in fact weighed by the squared correlation between money and output shocks). The calibrations also produce an estimated inflation premium on nominal bonds that, with one exception, varies between -50 and 30 basis points. These values appear to be reasonable in absolute values, since the ratio between the sample standard deviations of the ex-ante real rate and equity prices is approximately 1/10, which compares to a similar or lower ratio for bond and equity premia.

The scant available evidence on the US inflation risk premium suggests that this should be small, but positive at short maturities. For example, Buraschi and Jiltsov (2005) estimate an average, instantaneous inflation risk premium of approximately 25 basis points. This appears to speak in favour of the simulation based on the $\delta = 4$, $\gamma = 0.5$ assumption. Nevertheless, it is interesting to note that a negative inflation premium helps to explain the equity premium puzzle since, by construction, it is subtracted from the equity premium proper to obtain the relative equity premium.

In the case with a positive inflation risk premium, the natural rate is estimated to be around 1.1 percent, i.e. 0.4 percentage points lower than the observed, average real rate. If used in a simple policy rule, this would translate one-to-one into a systematically more expansionary policy stance than that produced by a rule based on the real rate.

5.2 The contribution of monetary uncertainty

In the rest of Table 2, I focus on the $\delta = 4$ case and analyze the effects of a 20 percent fall in ϕ_M, an exercise aiming to capture a scenario consistent with the period of "monetary instability" of the seventies.

Table 2 shows that the increased monetary uncertainty can easily generate sizable effects on the relative equity premium and on the observed real rate without appreciable changes in the corresponding worst case growth scenario. The ex-ante real rate falls by up to a whole percentage point and the relative equity premium increases by around three quarters of a percentage point. At the same time, the natural rate falls by an average of 0.2 percentage points. The variation in the relative equity premium is mostly due to a corresponding fall in the inflation premium, which turns negative in both cases taken into account.

The parallel increase in the relative equity premium and fall in the real rate may also be seen as consistent with the evidence of the seventies, if these are interpreted as a regime of monetary instability. During the seventies, the increase in the equity premium documented in Blanchard (1993) was accompanied by a prolonged fall of ex-ante real rates. This fall was particularly pronounced for short-term rates – which reached negative levels – but it was also observed at longer maturities.

5.3 Recent trends

Table 3 and 4 present some suggestive evidence on the recent evolution of the natural rate in the US and in the euro area. More specifically, Table 3 focuses on the post-1984 period, which could be seen as coincident with a regained monetary stability in the United States.[10] It then derives model-consistent natural rate and bond premia based on the sample values of the moments of the exogenous variables, the ex-post real rate and the Fama and French (2002) estimates of the post-war equity premium. For comparative purposes, Table 4 replicates the same exercise for the euro area based on the 1997-2001 sample and on the same value of the equity premium as in the US.[11]

Table 3 appears to confirm that the post-1984 data are consistent with the hypothesis of increased confidence in the monetary policy rule. The value of ϕ_M does in fact increase by approximately 150 percent. At the same time, the model interprets the data as consistent with a fall of confidence, though smaller in percentage terms, in the reference model of technology growth. This is necessary to explain the sizable increase in both the natural rate and the ex-ante real rate. The former appears to hover around the 3 percent level.

Euro area data appear to be consistent with a similar level of ambiguity of the reference model of technology growth, but much larger confidence in monetary stability. Almost identical values of the ex-ante real rate and the equity premium are consistent with a much higher value of the ϕ_M parameter. Consequently, the natural rate appears to be lower than in the US and approximately equal to 2.5 percent. The inflation premium also turns out to be closer to zero. The euro area results should obviously be interpreted with special caution, given the particularly short sample on which they are based.

6 Conclusions

This paper presented a model which analyses the importance of risk and uncertainty premia in determining the natural interest rate. I demonstrate that agents' confidence in the monetary policy rule followed by the central bank plays a nonnegligible role in shaping *real* premia, such as the premium incorporated in the natural rate and the equity premium.

An important note of caution is that these results are based on a highly stylized model and that the empirical calibration relies on a high degree of investors'

[10] In the more recent US sample, the standard deviations of the rates of growth of money M2 and real consumption and their correlation coefficient are $\sigma_M = 1.58$, $\sigma_y = 0.78$ and $\rho_{My} = 0.40$, respectively.

[11] In the 1980-2001 euro area sample, the standard deviations of the rates of growth of money M2 and real consumption and their correlation coefficient are $\sigma_M = 2.29$, $\sigma_y = 1.47$ and $\rho_{My} = 0.33$, respectively.

pessimism on the future outlook for consumption growth. Nevertheless, the paper highlights that standard estimates of the equilibrium natural rate of interest may be biased upwards. Given the important role played by the natural rate in many models, this bias can have important repercussions for their policy implications.

Table 1: Natural rate and risk premia under a single source of misspecification
United States
(in percentage points)

δ	γ	$\phi \times 100$	r^e	r	REP	EP	IP	WCG
3	564	∞	*1.5*	-1.2	2.4	5.1	2.7	2.5
3	0.5	0.140	*1.5*	-5.4	6.0	12.8	6.8	-10.3
3	6.0	0.322	*1.5*	-1.5	2.6	5.6	3.0	-3.1
4	564	∞	*1.5*	-1.2	2.4	5.1	2.7	2.5
4	0.5	0.104	*1.5*	-7.8	8.2	17.4	9.3	-14.9
4	6.0	0.304	*1.5*	-1.7	2.8	6.0	3.2	-3.4

Source: Campbell (2002) and Federal Reserve Board. Sample: 1959Q1-1999Q4.

Legend: δ: time preference rate (in percentage points); γ: coefficient of relative risk aversion; ϕ: agents' confidence in the reference model; r^e: real interest rate; r: natural interest rate – equation (7); REP: relative equity premium – equation (11); EP: equity premium; IP: inflation premium; WCG: implicit worst case consumption growth scenario.

Note: r^e matches by construction the sample mean of the ex-ante real interest rate through a suitable choice of ϕ.

Table 2: Natural rate and risk premia under two sources of misspecification
United States
(in percentage points)

δ	γ	$\phi_y \times 100$	$\phi_M \times 100$	r^e	REP	r	EP	IP	WCG
Benchmark calibration									
3	0.5	0.048	0.273	*1.5*	*5.2*	-2.9	9.5	4.3	-7.0
3	6.0	0.108	0.066	*1.5*	*5.2*	2.0	4.6	-0.5	-2.0
4	0.5	0.087	0.076	*1.5*	*5.2*	1.1	5.5	0.3	-3.0
4	6.0	0.098	0.070	*1.5*	*5.2*	1.6	5.0	-0.2	-2.4
Monetary instability scenario									
4	0.5	0.087	0.061	0.7	6.0	1.0	5.6	-0.4	-3.1
4	6.0	0.098	0.056	0.5	5.9	1.3	5.1	-0.8	-2.5

Source: Campbell (2002) and Federal Reserve Board. Sample: 1959Q1-1999Q4.

Legend: δ: time preference rate (in percentage points); γ: coefficient of relative risk aversion; ϕ_y and ϕ_M represent agents' confidence in the reference model of technology and money growth, respectively; r^e: real interest rate; r: natural interest rate – see equation (7); REP: relative equity premium – equation (11); EP: equity premium; IP: inflation premium; WCG: implicit worst case consumption growth scenario.

Note: r^e and REP match the sample means of the ex-ante real rate and the relative equity premium through a suitable choice of ϕ_y and ϕ_M; in the monetary instability scenario, ϕ_M is reduced by 20%.

Table 3: Recent low-equity-premium scenario
United States
(in percentage points)

δ	γ	$\phi_y \times 100$	$\phi_M \times 100$	r^e	REP	r	EP	IP	WCG
3	6.0	0.091	0.155	*2.5*	*4.3*	3.2	3.6	−0.7	−1.4
4	6.0	0.080	0.206	*2.5*	*4.3*	2.8	4.0	−0.3	−1.8

Source: Campbell (2002) and Federal Reserve Board. Sample: 1984Q1-1999Q4.

Legend: δ: time preference rate (in percentage points); γ: coefficient of relative risk aversion; ϕ_y and ϕ_M represent agents' confidence in the reference model of technology and money growth, respectively: r^e: real interest rate; r: natural interest rate – see equation (7); REP: relative equity premium – equation (11); EP: equity premium; IP: inflation premium; WCG: implicit worst case consumption growth scenario.

Note: r^e and REP match the sample mean of the ex-ante real rate and the largest estimate of the equity premium in Fama and French (2002), through a suitable choice of ϕ_y and ϕ_M.

Table 4: Recent low-equity-premium scenario
Euro area
(in percentage points)

δ	γ	$\phi_y \times 100$	$\phi_M \times 100$	r^e	REP	r	EP	IP	WCG
3	6.0	0.079	0.583	*2.4*	*4.3*	2.7	4.0	−0.3	−1.7
4	6.0	0.071	3.570	*2.4*	*4.3*	2.3	4.4	0.1	−2.1

Source: Fagan, Henry and Mestre (2001), ECB. Sample: 1997Q1-2001Q4.

Legend: δ: time preference rate (in percentage points); γ: coefficient of relative risk aversion; ϕ_y and ϕ_M represent agents' confidence in the reference model of technology and money growth; r^e: real interest rate; r: natural interest rate – see equation (7); REP: relative equity premium – equation (11); EP: equity premium; IP: inflation premium; WCG: implicit worst case consumption growth scenario.

Note: r^e and REP match the sample mean of the ex ante real rate and the same equity premium as in the US by suitable choice of ϕ_y and ϕ_M.

A Appendix

A.1 The model in the main text

The state vector of the model in the text is represented by the exogenous endowment and money growth processes (3) and (4), plus the endogenous process followed by real wealth.

I assume that households are not entirely sure about the above specification of endowment and money growth. In terms of the Uppal and Wang (2003) set-up, I define perturbations of the state vector $\mathbf{v} = (v_w, v_y, v_M)$, a vector $\sigma_X = (\sigma_w w, \sigma_y k, \sigma_M M)'$ such that the variance covariance matrix is

$$\sigma_X \sigma_X' = \begin{pmatrix} \sigma_w^2 w^2 & \sigma_{yw} kw & \sigma_{Mw} Mw \\ \sigma_{yw} kw & \sigma_y^2 k^2 & \sigma_{My} kM \\ \sigma_{Mw} Mw & \sigma_{My} kM & \sigma_M^2 M^2 \end{pmatrix}$$

and a matrix Φ capturing the level of ambiguity in the reference model. Notice that the wealth process is not subject to an independent source of misspecification, but only inherits misspecification through its components (whose returns are endogenously affected by potential misspecification in the state vector). The (also endogenous) correlation of real wealth with the endowment and money growth processes determines the extent to which misspecification in these state variables matters for the wealth process. It follows that perturbation $v_w = 0$ and ambiguity is captured by the matrix Φ defined in equation (5). In the special case of a single source of model misspecification, Φ is indexed by a single scalar ϕ such that $\Phi = \phi \cdot \begin{pmatrix} \sigma_y^2 y^2 & \sigma_{My} My \\ \sigma_{My} My & \sigma_M^2 M^2 \end{pmatrix}$.

The five assets available in the economy have returns defined as follows

$$\begin{aligned} \frac{\mathrm{d}s}{s} &= \left(\mu_s - \frac{y}{s}\right) \mathrm{d}t + \sigma_s \, \mathrm{d}z_s \\ \frac{\mathrm{d}M}{M} &= \left(-\mu_P + \sigma_P^2\right) \mathrm{d}t - \sigma_P \, \mathrm{d}z_P \\ \frac{\mathrm{d}B}{B} &= \left(R - \mu_P + \sigma_P^2\right) \mathrm{d}t - \sigma_P \, \mathrm{d}z_P \\ \frac{\mathrm{d}b}{b} &= r \, \mathrm{d}t \\ \frac{\mathrm{d}x}{x} &= \left(\mu_x - \frac{\tau}{x}\right) \mathrm{d}t + \sigma_x \, \mathrm{d}z_x \end{aligned}$$

Note that: the returns on equity, s, and on the present discounted value of future transfers, x, include a dividend yield as well as a capital gain; the returns on money and the nominal bond are zero and $R\mathrm{d}t$, respectively, in nominal terms, and negatively affected by inflation in real terms. In the equations above, R, r, the ratios $\frac{y}{s}$ and $\frac{\tau}{x}$ and all μ_i's and σ_i's have to be determined in equilibrium.

If P denotes the price of the consumption good, real wealth can be written as $w = s + M/P + B/P + b + x$ or, in terms of portfolio shares ω_i in asset i ($\omega_i \equiv \frac{i}{w}$ and $\sum_i \omega_i = 1$), $1 = \omega_s + \omega_B + \omega_b + \omega_x + \omega_M$. If I impose the constraint

$\omega_b = 1 - \omega_s - \omega_M - \omega_B - \omega_x$, the wealth process can be written as

$$\begin{aligned} \frac{\mathrm{d}w}{w} &= \left[(\omega_M + \omega_B)\left(R - \mu_P + \sigma_P^2 - r\right) + \omega_s(\mu_s - r) + \omega_x(\mu_x - r) - \omega_M R + r - \frac{c}{w} \right] \mathrm{d}t \\ & - (\omega_M + \omega_B)\sigma_P \mathrm{d}z_P + \omega_s \sigma_s \mathrm{d}z_s + \omega_x \sigma_x \mathrm{d}z_x \end{aligned} \tag{12}$$

or $\mathrm{d}w = \mu_w w \mathrm{d}t + \sigma_w w \mathrm{d}z_w$ using shorthand notation.

The investor's problem is then to

$$\max \int_0^\infty e^{-\delta t} \frac{u(c_t, m_t)^{1-\gamma}}{1-\gamma} \mathrm{d}t \qquad \gamma \neq 1$$

subject to the wealth accumulation process (12) while taking model misspecification into account.

Following Uppal and Wang (2003), the maximum value function of the optimization problem of the representative household can be written as a function $V = V(w_t, k_t, M_t, t)$ which has to solve

$$\begin{aligned} 0 &= \sup_{c,\boldsymbol{\omega}} \inf_{v_y, v_M} \Big\{ u(c) - \delta V + V_t + V_w \mu_w w + k \mu_y V_y + M \mu_M V_M \\ & + \frac{1}{2} V_{ww} \sigma_w^2 w^2 + V_{yw} \sigma_{wy} w y + V_{Mw} \sigma_{Mw} M w + \frac{1}{2} V_{yy} \sigma_y^2 y^2 \\ & + V_{My} \sigma_{My} M y + \frac{1}{2} V_{MM} \sigma_M^2 M^2 + V_w w (v_y \sigma_{yw} y + v_M \sigma_{Mw} M) \\ & + V_y y \left(v_y \sigma_y^2 y + v_M \sigma_{My} M\right) + V_M M \left(v_y \sigma_{My} y + v_M \sigma_M^2 M\right) \\ & + \frac{1}{2} \psi(V) \Big(v_y^2 \sigma_y^2 y^2 \left(2\phi_y + \phi_{My}\right) \\ & \quad + v_M^2 \sigma_M^2 M^2 \left(\phi_{My} + 2\phi_M\right) + 2 v_M v_y \sigma_{My} y M \phi_{My} \Big) \Big\} \end{aligned} \tag{13}$$

where σ_w^2 is the variance of the rate of growth of wealth and σ_{iw} the covariance between shocks to wealth and to process i. The term $\psi(V)$ is the function which converts the penalty for taking into account the information provided by a certain perturbation of the state vector into units of utility. The particular functional form of $\psi(\cdot)$ is chosen for analytical convenience. I follow Maenhout (2004) and Uppal and Wang (2003) and choose $\psi(V)$ to be proportional to the value function V as $\psi(V) = (1-\gamma)V$.

Given the utility function postulated above and that the money supply and production processes are lognormal, I can guess that V will be independent of the state variables M and y and of time. All corresponding derivatives of the maximum value function can therefore be set to zero. Using the conjecture $V(w) = \kappa \frac{w^{1-\gamma}}{1-\gamma}$ and the form of the utility function in equation (2), the first order conditions of the problem can be written as

$$
\begin{aligned}
\frac{\partial}{\partial c} &: \alpha^{\gamma} c_t^{-\gamma} R^{\frac{\varepsilon(\gamma\varepsilon-1)}{\varepsilon-1}} \left(R\left(1-\alpha\right)+R^{\varepsilon}\alpha\right)^{\frac{1-\gamma\varepsilon}{\varepsilon-1}} = w^{-\gamma}\kappa \\
\frac{\partial}{\partial \omega_M} &: m = c\frac{1-\alpha}{\alpha}R^{-\varepsilon} \\
\frac{\partial}{\partial \omega_B} &: R-\mu_P+\sigma_P^2-r = -\gamma\sigma_{Pw}+v_y y\sigma_{Py}+v_M M\sigma_{MP} \\
\frac{\partial}{\partial \omega_s} &: \mu_s - r = \gamma\sigma_{sw} - v_y y\sigma_{sy} - v_M M\sigma_{Ms} \\
\frac{\partial}{\partial \omega_x} &: \mu_x - r = \gamma\sigma_{wx} - v_y y\sigma_{xy} - v_M M\sigma_{Mx} \\
\frac{\partial}{\partial v_y} &: \gamma\sigma_{wy} + v_y\sigma_y^2 y\left(2\phi_y+\phi_{My}\right) + v_M\sigma_{My}M\phi_{My} = 0 \\
\frac{\partial}{\partial v_M} &: \gamma\sigma_{Mw} + v_M\sigma_M^2 M\left(\phi_{My}+2\phi_M\right) + v_y\sigma_{My}y\phi_{My} = 0.
\end{aligned}
$$

Postulate now that all covariances will have constant correlation coefficients, so that $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$ with ρ_{ij} being the correlation coefficient and σ_i and σ_j denoting standard deviations.

The 3rd to 5th first order conditions are asset pricing equations. For a generic asset i, one would obtain

$$\mu_i = r + \gamma\,\rho_{iw}\,\sigma_i\,\sigma_w - v_y\,y\,\rho_{iy}\,\sigma_i\,\sigma_y - v_M\,M\,\rho_{iM}\,\sigma_i\,\sigma_M \tag{14}$$

Now solve the macroeconomic side of the model. Recall that, in equilibrium, $c = y$ and that the total return on equity includes both a capital gain and a dividend, i.e. $\mathrm{d}s/s = (\mu_s - c/s)\mathrm{d}t + \sigma_s \mathrm{d}z_s$. The same holds true for the sum of real money holdings and the x asset, which can be defined as $\overline{x} \equiv x + m$, which yields the expected return μ_x plus the opportunity cost of holding money, equal to the foregone interest Rm, so that $\mathrm{d}\overline{x}/\overline{x} = (\mu_x - Rm/\overline{x})\mathrm{d}t + \sigma_x \mathrm{d}z_x$.

Conjecture that the evolution of real wealth is independent of monetary factors. Since nominal and real bonds are in zero net supply, equilibrium wealth is given by $w = s + \overline{x}$. It follows that $\mathrm{d}w/w = \mathrm{d}s/s = \mathrm{d}\overline{x}/\overline{x}$. Matching drift and diffusion terms, $(\mu_s - c/s) = (\mu_x - Rm/x)$ and $\sigma_s \mathrm{d}z_s = \sigma_x \mathrm{d}z_x = \sigma_w \mathrm{d}z_w$. Using the latter equality in the asset pricing equations for equity and the x asset, it follows that $\mu_s = \mu_x$, so that $c/s = Rm/\overline{x}$ and, rearranging terms, $\overline{x}/s = Rm/c = R^{1-\varepsilon}\left(1-\alpha\right)/\alpha$. From the equilibrium composition of real wealth, it finally follows that $w = \left(1 + R^{1-\varepsilon}\left(1-\alpha\right)/\alpha\right)s$. This can be used to substitute out w in the first order conditions and to obtain

$$c = R^{-\frac{\varepsilon(1-\gamma)}{\gamma(1-\varepsilon)}}\left(R\left(1-\alpha\right)+R^{\varepsilon}\alpha\right)^{-\frac{1-\gamma}{\gamma(1-\varepsilon)}}\kappa^{-\frac{1}{\gamma}}s \tag{15}$$

so that

$$\mu_w = \mu_s - R^{-\frac{\varepsilon(1-\gamma)}{\gamma(1-\varepsilon)}}\left(R\left(1-\alpha\right)+R^{\varepsilon}\alpha\right)^{-\frac{1-\gamma}{\gamma(1-\varepsilon)}}\kappa^{-\frac{1}{\gamma}} \tag{16}$$

The values of v_M, v_y, μ_w, and the second moments of wealth can be substituted back in the Bellman equation to verify that the conjectured form of the maximum value function does solve the equation for a constant value of κ. This value can be substituted in equations (15) and (16) so as to express them as functions of exogenous variables and first and second moments of the evolution of equity, s. Finally, noting that c is proportional to s, so that $\mathrm{d}c/c = \mathrm{d}s/s$, but also $\mathrm{d}c/c = \mathrm{d}y/y$ because of the equilibrium condition $c = y$, it follows that $\mathrm{d}s/s = \mathrm{d}y/y$. Hence $\sigma_s = \sigma_y$, $\rho_{sy} = 1$, $\rho_{Ms} = \rho_{My}$ and equation (16) must equal μ_y, which implies $\mu_S = \delta + \gamma\mu_y + (1-\gamma)\left(\gamma + \Phi_{My}^{-1} + \Phi_y^{-1} + \rho_{My}^2\Phi_M^{-1}\right)\frac{\sigma_y^2}{2}$, where Φ_{My}^{-1}, Φ_y^{-1} and Φ_M^{-1} are defined in the text – see equation (6).

The compounded coefficients Φ_{My}, Φ_y, Φ_M show how the various sources of model uncertainty are priced in the model (recall equation (6)). To understand why, note that the inverse of matrix Φ can be written as

$$\Phi^{-1} = \Phi_{My}^{-1}\begin{pmatrix} \frac{1}{(1-\rho_{My}^2)\sigma_y^2 y^2} & -\frac{\rho_{My}}{(1-\rho_{My}^2)\sigma_M\sigma_y My} \\ -\frac{\rho_{My}}{(1-\rho_{My}^2)\sigma_M\sigma_y My} & \frac{1}{(1-\rho_{My}^2)\sigma_M^2 M^2} \end{pmatrix} + \Phi_y^{-1}\begin{pmatrix} \frac{1}{\sigma_y^2 y^2} & 0 \\ 0 & 0 \end{pmatrix} + \Phi_M^{-1}\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\sigma_M^2 M^2} \end{pmatrix}$$

Hence Φ_{My}^{-1} prices the correlation between uncertainties over estimates of technology and money growth, while Φ_y^{-1} and Φ_M^{-1} price the marginal uncertainty of the estimates of the technology and money growth processes, respectively.

Using the equilibrium value of its return μ_S in the asset pricing equation for equity, and solving for r one finds

$$r = \delta + \gamma\mu_y - (1+\gamma)\left(\gamma + \frac{1}{\Phi_{My}} + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right)\frac{\sigma_y^2}{2}$$

that is Proposition 1. It follows that the equity premium is simply $EP = \left(\gamma + \frac{1}{\Phi_{My}} + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right)\sigma_y^2$, as in Proposition 3.

For the inflation rate, note that $Rm/c = R^{1-\varepsilon}(1-\alpha)/\alpha$ implies that, applying Ito's lemma,

$$\frac{dP}{P} = \left(\mu_M - \mu_y + \sigma_y^2 - \rho_{My}\sigma_M\sigma_y\right)dt + \sigma_M dz_M - \sigma_y dz_y \tag{17}$$

Defining the inflation target as minus the drift of the inverse of the price level $1/P$, i.e. $\mathrm{E}[\pi^*] = \mu_M - \mu_y - \sigma_M^2 + \rho_{My}\sigma_M\sigma_y$, it follows that this can be attained by an appropriate choice of the average rate of growth of money: $\mu_M^* = \mathrm{E}[\pi^*] + \mu_y + \sigma_M^2 - \rho_{My}\,\sigma_M\,\sigma_y$. Using equation (17) in the asset pricing equation for nominal bonds, it is finally possible to obtain the nominal interest rate

$$R = \mathrm{E}[\pi^*] + \delta + \gamma\mu_y - \frac{\gamma-1}{2}EP - \left(\gamma + \frac{1}{\Phi_y} + \frac{1}{\Phi_{My}} + \frac{1}{\Phi_M}\right)\rho_{My}\,\sigma_M\,\sigma_y$$

Using the equilibrium relationships, one can also find

$$\kappa^{\frac{1}{\gamma}} = R^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}} (1-\alpha)^{\frac{(1-\alpha)(1-\gamma)}{\gamma}} \alpha^{-\alpha\left(1-\frac{1}{\gamma}\right)} \frac{1}{\delta - (1-\gamma)\left(\mu_y - \frac{1}{2}EP\right)}$$

and

$$\begin{aligned} v_y y &= -\frac{1}{\Phi_y} + \frac{1}{\Phi_{My}} \\ v_M M &= -\frac{1}{\Phi_M}\frac{\rho_{My}\sigma_y}{\sigma_M} \end{aligned}$$

which allows me to rewrite equation (14) as equation (6) in the text.

The maximum value function is therefore given by

$$V = w^{1-\gamma}\left(\frac{1-\alpha}{R}\right)^{(1-\gamma)(1-\alpha)} \alpha^{\alpha(1-\gamma)} \frac{\left(\delta - (1-\gamma)\left(\mu_y - \frac{1}{2}EP\right)\right)^{1-\gamma}}{1-\gamma}$$

Feasibility of the optimal plan implies that optimum consumption $c(t) \geq 0$, which requires $\delta \geq (1-\gamma)\left(\mu_y - \frac{1}{2}EP\right)$.

Given the form of the maximum value function, $V = \frac{w^{1-\gamma}}{1-\gamma}\kappa$, the sufficient condition for the maximum, $\frac{\partial^2 V}{\partial w^2} = -\gamma w^{-1-\gamma}\kappa > 0$, requires $\kappa > 0$, which is always satisfied when the optimal plan is feasible.

The transversality condition $\lim_{t\to\infty} \mathrm{E}\left[\exp(-\delta t) V\right] = 0$ for a feasible plan requires $\lim_{t\to\infty} \mathrm{E}\left[e^{-\delta t} w^{1-\gamma}\right] = 0$. Given the equilibrium rate of growth of wealth, this limit tends to 0 as t tends to infinity provided that

$$\delta - (1-\gamma)\left(\mu_y - \frac{1}{2}EP\right) > \left(1 - \frac{1}{\gamma}\right)\left(1 - \left(\gamma + \frac{1}{\Phi_y} + \frac{1}{\Phi_{My}} + \frac{\rho_{My}^2}{\Phi_M}\right)\right)\frac{\sigma_y^2}{2}$$

The left hand side is always positive, by virtue of the feasibility condition. A sufficient condition for the transversality condition to be satisfied is therefore that the right had side of the equation is negative. When $\gamma > 1$, thus $\frac{\gamma-1}{\gamma} > 0$, it follows that $1 - \left(\gamma + \frac{1}{\Phi_y} + \frac{1}{\Phi_{My}} + \frac{\rho_{My}^2}{\Phi_M}\right) < 0$ and and the transversality condition is always satisfied. When $\gamma < 1$, $\frac{\gamma-1}{\gamma} < 0$ and the sign of $1 - \left(\gamma + \frac{1}{\Phi_y} + \frac{1}{\Phi_{My}} + \frac{\rho_{My}^2}{\Phi_M}\right)$ is likely to be negative for high levels of ambiguity in the reference model, so the sufficient condition may not be satisfied.

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